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Multilayer shells: Geometrically-exact formulation of equations of motion

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Abstract

We present here a formulation for geometrically-exact multilayer shells, where the number of layers is completely arbitrary and not necessarily restricted to three as in the case of sandwich shells. Based entirely in terms of stress resultants, the formulation accommodates finite deformation in membrane, bending, and shear deformation. Typical of geometrically-exact formulation, the kinematics of the shell is referred directly to an inertial frame, and not by means of a floating frame. The motion of a transverse fiber of an $\mathcal{N}$-layer shell is exactly that of a chain of $\mathcal{N}$ rigid links connected by universal joints, in three-dimensional space. The kinematics of deformation in each layer is expressed in terms of the deformation of an arbitrary layer chosen as a reference layer. The thickness and side dimensions of each layer are also arbitrary. These features allow a convenient modeling of multilayer shells with ply drop-offs. We also show that the present formulation reduces exactly to the case of multilayer beams. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Geometrically-exact shell theory; Composite multilayer shells; Layerwise formulation; Dynamics; Equations of motion; Large deformation; Large overall motion

1. Introduction

Multilayer structures have widespread applications in engineering, and have been a subject of investigation for several decades. A particular case of multilayer shells is the case of sandwich shells, which formed the central theme in the basic monograph of Plantema (1966). More recently, there are some review papers on formulations for multilayer plates (Noor and Burton, 1989) and more particularly for sandwich plates and shells (Noor et al., 1996). The readers are referred to these review

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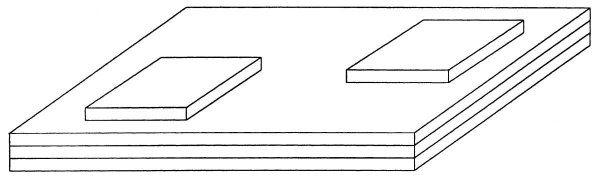

Fig. 1. Multilayer shells with patches of constrained viscoelastic materials or of piezoelectric materials.

papers and to those cited in our previous works[1] for a florilegium of references on these subjects. In Vu-Quoc et al. (1997b), we present the formulation for geometrically-exact sandwich shells that can undergo large deformation and large overall motion. In the present paper, we address the general case of geometrically-exact *multilayer* shells.

Several authors have considered the formulation of multilayered shells, with different approaches, and with a general limitation to the static case; the general dynamic case is considered in this paper. Pinsky and Kim (1986) employed the *degenerated-solid* approach to develop a multilayered-shell formulation that can account for small strain, large rotations, and elastic–viscoelastic 3D constitutive laws. Braun et al. (1994) extended the Reissner–Mindlin formulation from five parameters (no-drill degree of freedom) to seven parameters by using the Enhanced Assumed Strain approach, in which the transverse normal strain was allowed to vary linearly across the thickness of each layer. Similar to Pinsky and Kim (1986), complete 3D constitutive laws are employed in this formulation. Basar et al. (1993) generalized the formulation propounded by Reddy (1989), in which a cubic displacement field was employed, to the nonlinear range where finite rotation was allowed. The transverse director was originally normal to the shell mid-surface, as in Naghdi (1972). This formulation was restricted to the static case; the equations of equilibrium in terms of stress resultants were not derived, since the emphasis was on a finite element formulation. In fact, most prior works on nonlinear multilayer shells were primarily preoccupied with the static case, ignoring the dynamic aspects. Our formulation starts from a different perspective, from which the complete set of equations of motion for the dynamic case are derived. The computational aspects of our formulation are discussed elsewhere, in different papers. We refer to Vu-Quoc and Deng (1995, 1997a) for the static and dynamic computation of sandwich beams, and to Vu-Quoc et al. (2000b,c) for the static and dynamic computation of sandwich shells. We also note that our treatment of finite rotations is also different from Basar et al. (1993).

The present work parallels that in Vu-Quoc and Ebcioğlu (1996) for geometrically-exact multilayer (planar) beams and 1D plates. In fact, the methodology and crucial results developed in that reference are employed here for geometrically-exact multilayer shells. Typical for geometrically-exact formulation, the dynamics of motion of the multilayer shells in the present formulation is referred to a fixed inertial frame. Large deformation and large overall motion are accommodated for. The transverse fiber in each layer, inextensible in the present formulation, is not required to remain normal to the centroidal surface of that layer after deformation, thus allowing for shear deformation in each layer. The transverse fiber across the whole multilayer shell deforms as a *chain of rigid links* that are connected to each other by *universal* joints. The continuity of the displacement is thus exactly enforced across the layer boundaries. The number of layers can be even or odd, and is completely unrestricted. The thickness and side dimensions of each layer are arbitrary. It is, therefore, possible to use the present formulation to model multilayer shells with ply drop-offs, or with patches of constrained viscoelastic materials or of piezoelectric materials (see Fig. 1). The deformation of each layer is related to that of a reference layer,

[1] For example, Vu-Quoc and Ebcioğlu (1995, 1996)

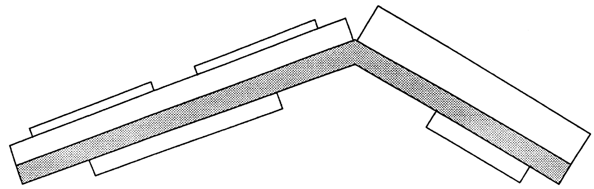

Fig. 2. Multilayer shells with arbitrary reference layer and with ply drop-offs.

chosen arbitrarily. This feature stands in sharp contrast to the formulation for sandwich shells in Vu-Quoc et al. (1997b), where the core (middle) layer was chosen as the reference layer. For example, in the present formulation, the top layer or the bottom layer can be chosen as the reference layer (see Fig. 2).

The equations of motion of the multilayer shell are derived based on the principle of virtual power, and expressed in terms of weighted resultants forces and couples. The principal kinematic quantities to be solved for are the deformation map of the centroidal surface of the reference layer (0), and the layer directors. Computationally, we have three translational components of the centroidal surface of the reference layer (0) and *two* rotational components for each layer director, since the layer directors are not rotating about themselves (no drill degree of freedom). We refer the readers to Vu-Quoc et al. (2000c) (statics) and Vu-Quoc et al. (2000b) (dynamics) for the computational formulations.

We show that the equations of motion for geometrically-exact shells developed herein reduce exactly to those of multilayer beams and 1D plates derived earlier in Vu-Quoc and Ebcioğlu (1996). Physical interpretation of the coupling terms that appear in the equations of motion is given. For truly large deformation in which the thickness of the shell is deformable, we refer the readers to Vu-Quoc and Ebcioğlu (2000a) for multilayer beams. We will report the result for multilayer shells with deformable thickness in a future paper.

2. Kinematics of deformation

The basic kinematics for geometrically-exact multilayer shells is presented in this section, together with some preliminary results that will be used in subsequent sections.

2.1. Basic kinematic assumptions and configurations

We present in Fig. 3, the profile of a multilayer shell in the material configuration $\mathscr{B}$. Let $\boldsymbol{\xi} \in \mathscr{B}$ designate a material point having the material coordinates $\{\xi^1, \xi^2, \xi^3\}$, and $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ be the associated basis vectors, as shown in Fig. 3. We denote the *reference layer* as layer (0); the kinematics of deformation of all other layers are referred to layer (0). In the present formulation, the number of layers in the shell is arbitrary, and so is the layer thickness. Any layer can be chosen as reference layer. Shown in Fig. 3 is an example of a five-layer shell, with the second layer from bottom chosen as the reference layer (0). Below, we will describe the notation adopted for various kinematic quantities.

Once the reference layer (0) is chosen, layers above Layer (0) are numbered with positive integers, and layers below Layer (0) with negative integers. Let $\ell \in \mathbb{Z}_+$ designate a layer number.[2] When both the

[2] $\mathbb{Z}_+ = \{0, 1, 2, 3, \ldots\}$ is the set of non-negative integers.

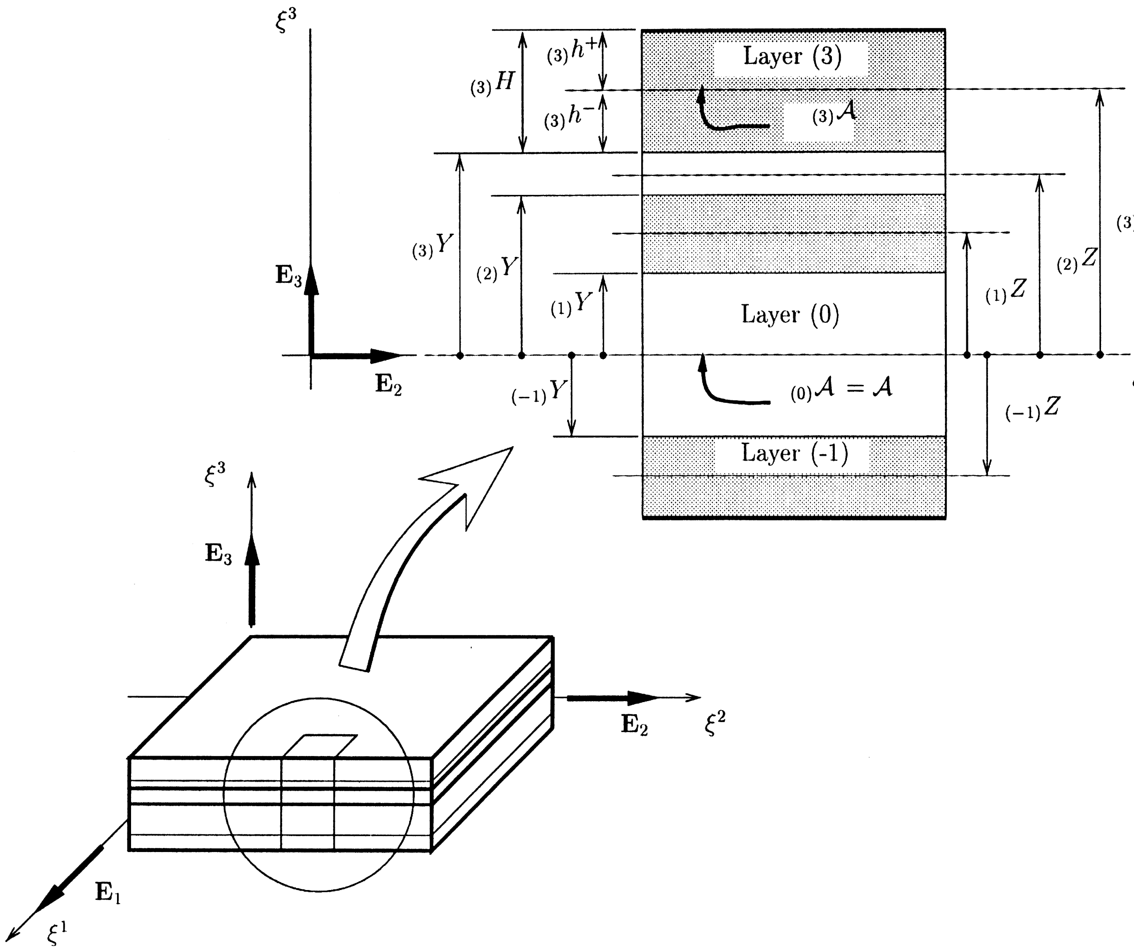

Fig. 3. *Multilayer shell*: Profile and geometric quantities.

upper layer $(+\ell)$ and the lower layer $(-\ell)$ are present, we often refer to both of these layers by $(s\ell)$, where $s = \pm 1$ designates the sign (positive for upper layers, and negative for lower layers). This compact notation will prove to be convenient for the exposition of the formulation.

Following Vu-Quoc and Ebcioğlu (1996), we let $\check{\mathscr{N}}$ be the number of lower layers, $\hat{\mathscr{N}}$ the number of upper layers, and

$$\underline{\mathscr{N}} := \{ -\check{\mathscr{N}}, \ldots, -1, 0, 1, \ldots, \hat{\mathscr{N}} \}, \tag{1}$$

$$\mathscr{N} := \check{\mathscr{N}} + 1 + \hat{\mathscr{N}}, \tag{2}$$

the set of layer labels and the total number of layers, respectively. Note that, in general, $\check{\mathscr{N}} \neq \hat{\mathscr{N}}$, i.e., the number of lower layers need not be the same as the number of upper layers. Our formulation thus allows for an arbitrary choice of the reference layer (0) to be anywhere from the bottom layer to the top layer. Let the origin of the transverse material coordinate ξ^3 be at the material centroidal surface [3] of layer (0), called the *material reference centroidal surface* ${}_{(0)}\mathscr{A} \equiv \mathscr{A}$, which is coordinatized by $\boldsymbol{\xi}^{\underline{\alpha}} := \{\xi^1, \xi^2\}$. The distance from ${}_{(0)}\mathscr{A}$ to the material centroidal surface ${}_{(s\ell)}\mathscr{A}$ of layer $(s\ell)$ is denoted by ${}_{(s\ell)}Z > 0$.[4] With ${}_{(\ell)}\rho_t$ being the mass density in the current configuration $\mathscr{B}_t$, the centroidal surface of layer $(s\ell)$ is then defined such that

$$\int_{{}_{(\ell)}\mathscr{H}} \left(\xi^3 - s_{(s\ell)}Z\right)_{(s\ell)} j_{t(s\ell)} \rho_t \, \mathrm{d}\xi^3 = 0, \quad \forall t, \forall s\ell \in \underline{\mathscr{N}}, \tag{3}$$

and does not in general coincide with the geometric center of that layer. It should be noted that the *overall* neutral surface of the multilayer shell, i.e., the surface on which the membrane stresses are zero, is *not* necessarily the centroidal surface of layer (0) (or the reference centroidal surface).

The surface ${}_{(s\ell)}\mathscr{A}$ is at the distance ${}_{(s\ell)}h^+$ from the top of layer $(s\ell)$, and at ${}_{(s\ell)}h^-$ from the bottom of layer $(s\ell)$. The thickness of layer $(s\ell)$ is given by

$${}_{(s\ell)}H := {}_{(s\ell)}h^+ + {}_{(s\ell)}h^- = {}_{(s\ell)}h^{+s} + {}_{(s\ell)}h^{-s}, \tag{4}$$

with ${}_{(s\ell)}h^+ \neq {}_{(s\ell)}h^-$ in general. In Eq. $(4)_2$, note that ${}_{(s\ell)}h^{-s} = {}_{(\ell)}h^-$ for $s = +1$, and ${}_{(s\ell)}h^{-s} = {}_{(-\ell)}h^+$ for $s = -1$; a similar interpretation holds for ${}_{(s\ell)}h^{+s}$. Further,

$${}_{(0)}Z = 0, \qquad {}_{(s\ell)}Z = \left[{}_{(s\ell)}Y + {}_{(s\ell)}h^{-s}\right], \quad \text{for } s\ell \in \underline{\mathscr{N}} \backslash \{0\}, \tag{5}$$

$${}_{(s\ell)}Y := -\,{}_{(0)}h^{-s} + \sum_{i=0}^{s(\ell-1)} {}_{(i)}H = {}_{(0)}h^{+s} + \sum_{i=s}^{s(\ell-1)} {}_{(i)}H, \quad \text{for } s\ell \in \underline{\mathscr{N}} \backslash \{0\}, \tag{6}$$

where ${}_{(s\ell)}Y$ designates the *distance* from the material reference centroidal surface of layer (0) to the interface between layer $(s[\ell-1])$ and layer $(s\ell)$; see Fig. 3.

Remark 2.1. *It is clear that when we write $\ell \in \underline{\mathscr{N}}$, the index ℓ designates both the upper and the lower layers; in cases like these when there is no confusion, we thus omit the use of the sign s. Thus, layer (ℓ) with $\ell \in \underline{\mathscr{N}}$ can also be designated as layer $(s\ell)$ with $s\ell = \mathrm{sign}(\ell)\,|\ell| \in \underline{\mathscr{N}}$, where sign is the signum function, and $|\cdot|$ the absolute value operator. In other words, in the notation $(s\ell)$, ℓ always takes on* positive *values.*

Remark 2.2. *In view of* Remark 2.1, *the summation in* Eq. $(6)_1$ *is to be interpreted as follows: In the upper summation limit $s(\ell-1)$, we have $(\ell-1) \geq 0$, and thus the sum in* Eq. $(6)_1$ *is to be carried out only when $\ell \geq 1$. There are two cases for* Eq. $(6)_1$: (i) *If $s = +1$ (i.e., upper layers), the range of the summation index i is* $\{0, 1, 2, \ldots, (\ell-1)\}$; (ii) *If $s = -1$ (i.e., lower layers), the range of i is* $\{0, -1, -2, \ldots, -(\ell-1)\}$. *Since for $\ell = 0$, there is no sum, it is clear that with this convention,* Eq. $(5)_1$ *is a particular case of* Eq. $(5)_2$, *after substitution of* Eq. $(6)_1$ *into* Eq. $(5)_2$. *Similarly, for* Eq. $(6)_2$, *the summation is carried out only for $\ell \geq 2$.*

Let ${}_{(\ell)}\mathscr{H} \ni \xi^3$ be the domain in the thickness direction of layer (ℓ), such that

[3] A precise definition of the *centroidal* surface will be given later in Eq. (3).

[4] Note that ${}_{(s\ell)}Z$ is not the ordinate, but the distance, and is thus a positive number.

$$_{(s\ell)}\mathscr{H} := \left[\left(s_{(s\ell)}Z - {}_{(s\ell)}h^{-} \right), \left(s_{(s\ell)}Z + {}_{(s\ell)}h^{+} \right) \right] \subset \mathbb{R}, \tag{7}$$

and $H := \cup_{\ell \in \underline{\mathscr{N}}} {}_{(\ell)}\mathscr{H}$ the domain in the thickness direction of the whole multilayer shell. Since the projection of all material centroidal surfaces $_{(\ell)}\mathscr{A}$, $\forall \ell \in \underline{\mathscr{N}}$, onto the plane $\{\xi^1, \xi^2\}$ is denoted by $\mathscr{A} \subset \mathbb{R}^2$, the material domain of layer (ℓ), denoted by $_{(\ell)}\mathscr{B}$, and the material configuration $\mathscr{B}$ can be expressed by

$$_{(\ell)}\mathscr{B} = \mathscr{A} \times {}_{(\ell)}\mathscr{H}, \quad \mathscr{B} = \mathscr{A} \times \mathscr{H} = \bigcup_{\ell \in \underline{\mathscr{N}}} {}_{(\ell)}\mathscr{B}. \tag{8}$$

Further, we define

$$_{(\ell)}\mathscr{S} := \partial\mathscr{A} \times {}_{(\ell)}\mathscr{H} \tag{9}$$

to be the material lateral surface of layer (ℓ), where $\partial\mathscr{A}$ is the boundary of $\mathscr{A}$; see Fig. 4. Then

$$\mathscr{S} = \bigcup_{\ell \in \underline{\mathscr{N}}} {}_{(\ell)}\mathscr{S} \tag{10}$$

is the (material) lateral surface of $\mathscr{B}$.

It follows from Eqs. (8) and (9) that the outward normal $_{(\ell)}\boldsymbol{v} = {}_{(\ell)}v_{\alpha}\mathbf{E}^{\alpha}$ to the material lateral boundary surface $_{(\ell)}\mathscr{S}$, which is defined in Eq. (9), is such that

$$_{(\ell)}\boldsymbol{v} = {}_{(0)}\boldsymbol{v}, \quad \forall \ell \in \underline{\mathscr{N}} \backslash \{0\}. \tag{11}$$

That is, the outward normals to the material lateral boundary surfaces $_{(\ell)}\mathscr{S}$ are the same for all layers. In other words, the surfaces $_{(\ell)}\mathscr{S}$ are all parallel to each other, and orthogonal to the material centroidal surface $\mathscr{A} \equiv {}_{(0)}\mathscr{A}$ (see Figs. 3 and 4).

Let the *initial configuration* of the multilayer shell be denoted by $\mathscr{B}_0 \subset \mathbb{R}^3$, and the *current (spatial) configuration* denoted by $\mathscr{B}_t \subset \mathbb{R}^3$ (see Fig. 5). Let $\boldsymbol{\Phi}_0 : \mathscr{B} \rightarrow \mathscr{B}_0$ be the deformation map from material

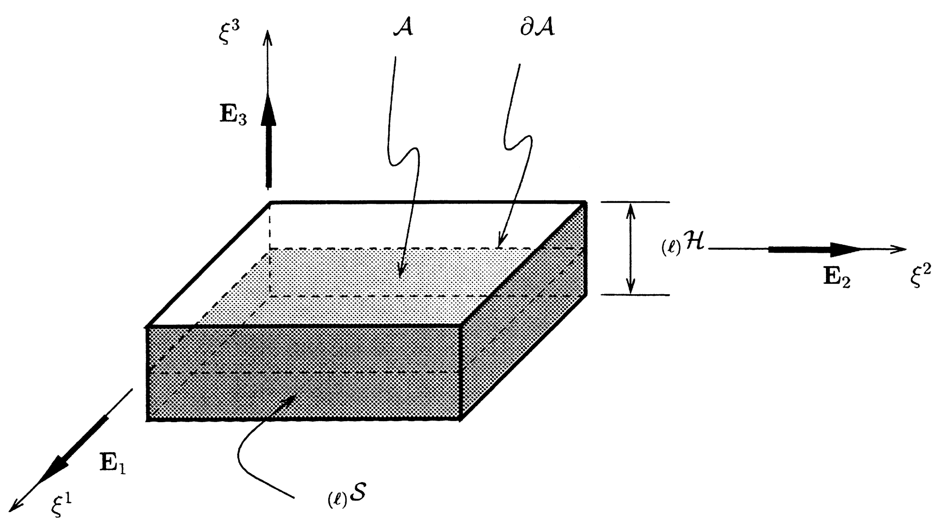

Fig. 4. Material configuration $_{(\ell)}\mathscr{B}$ of layer (ℓ): lateral surface $_{(\ell)}\mathscr{S}$.

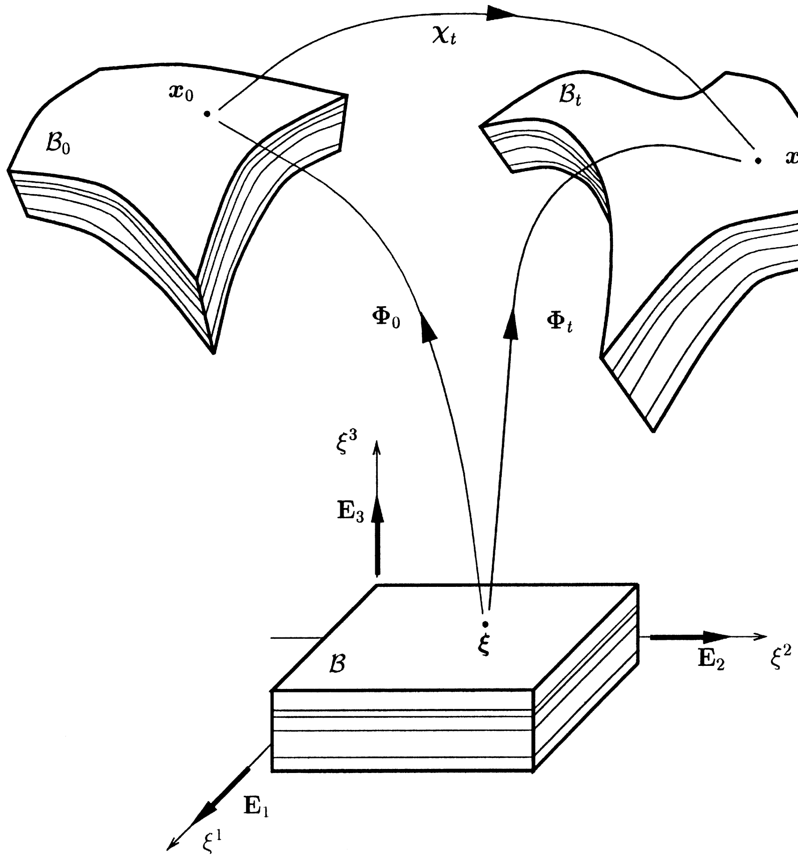

Fig. 5. *Multilayer shell*: material configuration $\mathscr{B}$, initial configuration $\mathscr{B}_0$, and current configuration $\mathscr{B}_t$.

configuration $\mathscr{B}$ to the initial configuration $\mathscr{B}_0$, such that $\boldsymbol{x}_0 = \boldsymbol{\Phi}_0(\boldsymbol{\xi}) \in \mathscr{B}_0$, where $\boldsymbol{\xi} \in \mathscr{B}$. Let $\boldsymbol{\Phi}_t: \mathscr{B} \rightarrow \mathscr{B}_t$ be the deformation map from $\mathscr{B}$ to the current configuration $\mathscr{B}_t$, such that $\boldsymbol{x}_t = \boldsymbol{\Phi}_t(\boldsymbol{\xi}) \in \mathscr{B}_t$, where $\boldsymbol{\xi} \in \mathscr{B}$. Further, we define the deformation map $\boldsymbol{\Phi}: \mathscr{B} \times \mathbb{R}_+ \rightarrow \mathbb{R}^3$, such that[5]

$$\boldsymbol{\Phi}_0(\,\cdot\,) \equiv \boldsymbol{\Phi}(\,\cdot\,, 0), \qquad \boldsymbol{\Phi}_t(\,\cdot\,) \equiv \boldsymbol{\Phi}(\,\cdot\,, t). \tag{12}$$

The deformation map for the shell relative to the initial configuration is denoted by $\boldsymbol{\chi}_t: \mathscr{B}_0 \rightarrow \mathscr{B}_t$ (see Fig. 5), with

$$\boldsymbol{\chi}_t = \boldsymbol{\Phi}_t \circ \boldsymbol{\Phi}_0^{-1}. \tag{13}$$

With ${}_{(\ell)}\boldsymbol{\Phi}_t: {}_{(\ell)}\mathscr{B} \rightarrow \mathbb{R}^3$ being the deformation map for layer (ℓ), we have

$$\boldsymbol{\Phi}_t(\boldsymbol{\xi}) = {}_{(\ell)}\boldsymbol{\Phi}_t(\boldsymbol{\xi}), \quad \forall \boldsymbol{\xi} \in {}_{(\ell)}\mathscr{B}, \tag{14}$$

and also

[5] $\mathbb{R}_+$ is the set of non-negative real numbers.

$$ {}_{(\ell)}\boldsymbol{\chi}_t = {}_{(\ell)}\boldsymbol{\Phi}_t \circ {}_{(\ell)}\boldsymbol{\Phi}_0^{-1}. \tag{15} $$

Now let ${}_{(\ell)}\boldsymbol{t}: \mathscr{A} \times \mathbb{R}_+ \to S^2$ be the director field of layer (ℓ), where S^2 designates the sphere defined as

$$ S^2 := \left\{ \boldsymbol{t} \in \mathbb{R}^3 \,|\, \|\boldsymbol{t}\| = 1 \right\}. \tag{16} $$

With ${}_{(\ell)}\boldsymbol{\varphi}: \mathscr{A} \times \mathbb{R}_+ \to \mathbb{R}^3$ being the deformation map of the centroidal surface of layer (ℓ), we can now define the deformation map ${}_{(\ell)}\boldsymbol{\Phi}_t$ in Eq. (14) as follows

$$ {}_{(s\ell)}\boldsymbol{\Phi}_t(\xi) = {}_{(s\ell)}\boldsymbol{\Phi}(\xi, t) := {}_{(s\ell)}\boldsymbol{\varphi}(\xi^{\underline{\alpha}}, t) + \left(\xi^3 - s_{(s\ell)}Z \right) {}_{(s\ell)}\boldsymbol{t}(\xi^{\underline{\alpha}}, t), \tag{17} $$

which basically describes that the deformed transverse fiber remains straight, colinear with the layer director ${}_{(s\ell)}\boldsymbol{t}$, and thus not necessarily perpendicular to the deformed centroidal surface ${}_{(\ell)}\boldsymbol{\varphi}_t(\mathscr{A}) \equiv {}_{(\ell)}\boldsymbol{\varphi}(\mathscr{A}, t)$ of layer (ℓ). Now, we describe the expression for the deformation map ${}_{(\ell)}\boldsymbol{\varphi}$ of the centroidal surface of layer (ℓ), such that continuity across all layers is strictly enforced:

$$ {}_{(s\ell)}\boldsymbol{\varphi} := {}_{(s[\ell-1])}\boldsymbol{\varphi}^{+s} + s_{(s\ell)}h^{-s}{}_{(s\ell)}\boldsymbol{t}, \quad \forall \xi^{\underline{\alpha}} \in \mathscr{A} \tag{18} $$

where

$$ {}_{(s[\ell-1])}\boldsymbol{\varphi}^{+s} := {}_{(0)}\boldsymbol{\varphi} + s\left[-{}_{(0)}h^{-s}{}_{(0)}\boldsymbol{t} + \sum_{i=0}^{s(\ell-1)} {}_{(i)}H_{(i)}\boldsymbol{t} \right]. \tag{19} $$

Remark 2.3. *First, assume that* $s = +1$ *in* Eq. (18), *and consider the positive layer* (ℓ) $(\ell > 0)$ *above the reference layer* (0). *In this case, the function* ${}_{([\ell-1])}\boldsymbol{\varphi}^{+}$ *in* Eq. (19) *represents the deformation map of the top surface of layer* $([\ell-1])$ *(i.e., the layer just below layer* (ℓ)*). On the other hand, for the negative layer* $(-\ell)$ *(i.e., for* $s = -1$ and $\ell > 0$), ${}_{(-[\ell-1])}\boldsymbol{\varphi}^{-}$ *represents the deformation map of the bottom surface of layer* $(-[\ell-1])$ *(i.e., the layer just* above *layer* $(-\ell)$*). Next, the construction of the expression* (19) *for* ${}_{(s[\ell-1])}\boldsymbol{\varphi}^{+s}$ *begins with the deformation map* ${}_{(0)}\boldsymbol{\varphi}$ *of the reference layer* (0), *and followed by the addition of the transverse fiber vectors (e.g.,* ${}_{(i)}H_{(i)}\boldsymbol{t}$*) of all layers between layer* (0) *and layer* $(s\ell)$. *For example, from* Eqs. (18) and (19), *the deformation map for layer* (s) *(i.e., layer* $(+1)$ *or layer* (-1) *(see* Fig. 3) *is described by*

$$ {}_{(s)}\boldsymbol{\varphi} = {}_{(0)}\boldsymbol{\varphi}^{+s} + s_{(s)}h^{-s}{}_{(s)}\boldsymbol{t}, \tag{20} $$

$$ {}_{(0)}\boldsymbol{\varphi}^{+s} = {}_{(0)}\boldsymbol{\varphi} + s_{(0)}h^{+s}{}_{(0)}\boldsymbol{t}. \tag{21} $$

Take another example; consider layer $(+3)$. *Combining the expressions* Eqs. (18) and (19), *we obtain*

$$ {}_{(3)}\boldsymbol{\varphi} = \left[{}_{(0)}\boldsymbol{\varphi} + {}_{(0)}h^{+}{}_{(0)}\boldsymbol{t} + \sum_{i=1}^{2} {}_{(i)}H_{(i)}\boldsymbol{t} \right] + {}_{(3)}h^{-}{}_{(3)}\boldsymbol{t}, \tag{22} $$

where we had used relation (4).

Next, we describe the director rotation map ${}_{(\ell)}\boldsymbol{\Lambda}: S^2 \to S^2$ that maps the material basis vector $\mathbf{E}_3$ to the director ${}_{(\ell)}\boldsymbol{t}$ of layer (ℓ). To describe where ${}_{(\ell)}\boldsymbol{\Lambda}$ belongs to, we define

$$ S_E^2 := \left\{ \boldsymbol{\Lambda} \in SO(3) \,|\, \boldsymbol{\Lambda\Psi} = \boldsymbol{\Psi}, \forall \boldsymbol{\Psi} \in \mathbb{R}^3 \text{ and } \boldsymbol{\Psi}\cdot\mathbf{E}_3 = 0 \right\}, \tag{23} $$

where $SO(3)$ is the special group of proper orthogonal transformations in $\mathbb{R}^3$. Thus, $\boldsymbol{\Lambda} \in S_E^2$ rotates $\mathbf{E}_3$ about axes of rotation perpendicular to $\mathbf{E}_3$, and thus induces no rotations about the $\mathbf{E}_3$ vector itself. We, therefore, eliminate the drilling degrees-of-freedom about the directors in this formulation. To each director field ${}_{(\ell)}\boldsymbol{t}$ associated with layer (ℓ), there exists a unique operator ${}_{(\ell)}\boldsymbol{\Lambda} \in S_E^2$ that maps $\mathbf{E}_3$ into ${}_{(\ell)}\boldsymbol{t}$, i.e.,

$${}_{(\ell)}\boldsymbol{t} = {}_{(\ell)}\boldsymbol{\Lambda}\cdot\mathbf{E}_3 \tag{24}$$

2.2. Principal unknown kinematic fields

Basically, the deformation of the multilayer shell can be described by the deformation map ${}_{(0)}\boldsymbol{\varphi}$ of the reference layer (0) and by the $\mathcal{N}$ director rotation maps ${}_{(\ell)}\boldsymbol{\Lambda}$, for $\ell \in \underline{\mathcal{N}}$. We thus have $(1+\mathcal{N})$ unknown functions. From the computational viewpoint, we need to introduce the displacement field $\tilde{\boldsymbol{u}}: \mathscr{B} \times \mathbb{R}_+ \rightarrow \mathscr{B}_t$

$$\tilde{\boldsymbol{u}}(\xi^{\underline{\alpha}}, t) := {}_{(0)}\boldsymbol{\varphi}(\xi^{\underline{\alpha}}, t) - \xi^{\underline{\alpha}}\mathbf{E}_\alpha \Longrightarrow {}_{(0)}\boldsymbol{\varphi}(\xi^{\underline{\alpha}}, t) = \xi^{\alpha}\mathbf{E}_\alpha + \tilde{\boldsymbol{u}}(\xi^{\underline{\alpha}}, t). \tag{25}$$

First, define the initial deformation map ${}_{(0)}\varphi_0: \mathscr{A} \rightarrow \mathscr{B}_0$ for the reference layer (0)

$${}_{(0)}\boldsymbol{\varphi}_0(\xi^{\underline{\alpha}}) := {}_{(0)}\boldsymbol{\varphi}(\xi^{\underline{\alpha}}, 0). \tag{26}$$

Similar to Eq. (25), the initial displacement field $\tilde{\boldsymbol{u}}_0: \mathscr{B} \rightarrow \mathscr{B}_0$

$$\tilde{\boldsymbol{u}}_0(\xi^{\underline{\alpha}}) := {}_{(0)}\boldsymbol{\varphi}_0(\xi^{\underline{\alpha}}) - \xi^{\alpha}\mathbf{E}_\alpha \Longrightarrow {}_{(0)}\boldsymbol{\varphi}_0(\xi^{\underline{\alpha}}) = \xi^{\alpha}\mathbf{E}_\alpha + \tilde{\boldsymbol{u}}_0(\xi^{\underline{\alpha}}). \tag{27}$$

Finally, the displacement field $\boldsymbol{u}: \mathscr{B}_0 \times \mathbb{R}_+ \rightarrow \mathscr{B}t$ that maps the reference layer (0) from the initial configuration $\mathscr{B}_0$ to the current configuration $\mathscr{B}_t$ is defined as

$$\boldsymbol{u} := \tilde{\boldsymbol{u}} - \tilde{\boldsymbol{u}}_0. \tag{28}$$

So, the principal unknown kinematic fields are the displacement field $\boldsymbol{u}$, and $\mathcal{N}$ director fields ${}_{(\ell)}\boldsymbol{t}$, with $\ell \in \underline{\mathcal{N}}$. In terms of components, we therefore have $3(1+\mathcal{N})$ unknown functions, with independent variables $\boldsymbol{\xi}^{\underline{\alpha}} \in \mathscr{A}$ and $\boldsymbol{t} \in \mathbb{R}_+$.

Remark 2.4. *It should be noted that even though the principal kinematic unknowns are* $\boldsymbol{u}$ *and* ${}_{(\ell)}\boldsymbol{t}$, *with* $\ell \in \underline{\mathcal{N}}$, *the computational kinematic unknowns are not the same. In the computation, we do not compute* ${}_{(\ell)}\boldsymbol{t}$ *directly, but rather the rotation vector* ${}_{(\ell)}\boldsymbol{\theta}$, *which then allows the evaluation of the rotation tensor* ${}_{(\ell)}\boldsymbol{\Lambda}$ *and then the computation of* ${}_{(\ell)}\boldsymbol{t}$ *following* Eq. (24); *we refer the reader to* Simo and Fox (1989) *for the single-layer case, and to* Vu-Quoc et al. (2000c) *for the multilayer case. Thus, the* computational kinematic unknowns *are* $\boldsymbol{u}$ *and* ${}_{(\ell)}\boldsymbol{\theta}$, *with* $\ell \in \underline{\mathcal{N}}$.

2.3. Space and time derivatives of kinematic fields

In this section, we gather the derivatives of the principal kinematic fields, which will be used in subsequent sections. We begin first with the space derivatives, followed by the time derivatives, and then the mixed space–time derivatives.

Let ${}_{(\ell)}\boldsymbol{g}_i$ be the convected basis vector, in the current configuration $\mathscr{B}_t$, along the convected coordinate ξ^i, and defined as

$$ {}_{(\ell)}\boldsymbol{g}_i(\xi, t) := \frac{\partial_{(\ell)}\boldsymbol{\Phi}(\xi, t)}{\partial \xi^i} =: {}_{(\ell)}\boldsymbol{\Phi}_{,i}(\xi, t), \quad \forall i \in \{1, 2, 3\}, \forall \ell \in \underline{\mathcal{N}}. \tag{29} $$

Next, we distinguish the case where $i = 1, 2$ and $i = 3$ in Eq. (29) to relate the convected basis vectors ${}_{(\ell)}\boldsymbol{g}_i$ to the principal kinematic fields defined earlier. Using Eqs. (17) and (29), we obtain

$$ {}_{(s\ell)}\boldsymbol{g}_\alpha = {}_{(s\ell)}\boldsymbol{\varphi}_{,\alpha} + \left(\xi^3 - s_{(s\ell)}Z\right)_{(s\ell)}\boldsymbol{t}_{,\alpha}, \quad \forall \alpha \in \{1, 2\}, \forall s\ell \in \underline{\mathcal{N}}, \tag{30} $$

$$ {}_{(\ell)}\boldsymbol{g}_3 = {}_{(\ell)}\boldsymbol{t}, \quad \forall \ell \in \underline{\mathcal{N}}. \tag{31} $$

The space derivative of ${}_{(s\ell)}\boldsymbol{\varphi}$ in Eq. (30) is then related to the principal kinematic fields by using Eqs. (18), (19) and (25) to obtain

$$ {}_{(0)}\boldsymbol{\varphi}_{,\alpha} = \mathbf{E}_\alpha + \tilde{\boldsymbol{u}}_{,\alpha}, \tag{32} $$

$$ {}_{(s\ell)}\boldsymbol{\varphi}_{,\alpha} = {}_{(s[\ell-1])}\boldsymbol{\varphi}^{+s}_{,\alpha} + s_{(s\ell)}h^{-s}{}_{(s\ell)}\boldsymbol{t}_{,\alpha}, \quad \forall \xi^{\underline{\alpha}} \in \mathcal{A}, \tag{33} $$

where

$$ {}_{(s[\ell-1])}\boldsymbol{\varphi}^{+s}_{,\alpha} = {}_{(0)}\boldsymbol{\varphi}_{,\alpha} + s\left[-{}_{(0)}h^{-s}{}_{(0)}\boldsymbol{t}_{,\alpha} + \sum_{i=0}^{s(\ell-1)} {}_{(i)}H_{(i)}\boldsymbol{t}_{,\alpha} \right]. \tag{34} $$

The convected basis vectors for layer (ℓ) in the initial configuration $\mathscr{B}_0$ are denoted by ${}_{(\ell)}\boldsymbol{G}_I$, and are the basis vectors ${}_{(\ell)}\boldsymbol{g}_I$ in Eq. (29) at $t = 0$

$$ {}_{(\ell)}\boldsymbol{G}_I(\xi) := {}_{(\ell)}\boldsymbol{g}_I(\xi, 0) = {}_{(\ell)}\boldsymbol{\Phi}_{,I}(\xi, 0) = {}_{(\ell)}\boldsymbol{\Phi}_{0,\,I}(\xi), \quad \forall I \in \{1, 2, 3\}, \forall \ell \in \underline{\mathcal{N}}, \tag{35} $$

where ${}_{(\ell)}\boldsymbol{\Phi}_0$ was defined in Eq. (12). The basis vectors dual to $\{{}_{(\ell)}\boldsymbol{g}_i\}$ and to $\{{}_{(\ell)}\boldsymbol{G}_I\}$ are defined by the orthogonality conditions

$$ \langle {}_{(\ell)}\boldsymbol{g}^i, {}_{(\ell)}\boldsymbol{g}_j \rangle = \delta^i_j, \qquad \langle {}_{(\ell)}\boldsymbol{G}^I, {}_{(\ell)}\boldsymbol{G}_J \rangle = \delta^I_J, \tag{36} $$

where $\langle \cdot, \cdot \rangle$ is the inner (dot) product in $\mathbb{R}^3$ and δ^i_j the Kronecker delta.

Remark 2.5. *It is noted that* ${}_{(\ell)}\boldsymbol{g}^3 \neq {}_{(\ell)}\boldsymbol{g}_3 \equiv {}_{(\ell)}\boldsymbol{t}$*, in general, such that we always have* ${}_{(\ell)}\boldsymbol{g}^3 \cdot {}_{(\ell)}\boldsymbol{g}_3 = 1$*, and* $\|{}_{(\ell)}\boldsymbol{g}^3\| \neq \|{}_{(\ell)}\boldsymbol{g}_3\| = \|{}_{(\ell)}\boldsymbol{t}\| = 1$

The gradient of the deformation map ${}_{(\ell)}\boldsymbol{\chi}_t$ in Eq. (15), with respect to the convected coordinates ξ, denoted by ${}_{(\ell)}\boldsymbol{F}$, is obtained as follows

$$ {}_{(\ell)}\boldsymbol{F} = \mathrm{GRAD}_{(\ell)}\boldsymbol{\chi}_t = \mathrm{GRAD}_{(\ell)}\boldsymbol{\Phi}_t \circ \left[\mathrm{GRAD}_{(\ell)}\boldsymbol{\Phi}_0\right]^{-1}. \tag{37} $$

In component form, we have

$$ \mathrm{GRAD}_{(\ell)}\boldsymbol{\Phi}_t = \delta^i_J\, {}_{(\ell)}\boldsymbol{g}_i \otimes \mathbf{E}^J, \qquad \left[\mathrm{GRAD}_{(\ell)}\boldsymbol{\Phi}_0\right]^{-1} = \delta^I_J \mathbf{E}_I \otimes {}_{(\ell)}\boldsymbol{G}^J. \tag{38} $$

It follows that with respect to the convected basis vectors, the deformation gradients ${}_{(\ell)}\boldsymbol{F}$ and ${}_{(\ell)}\boldsymbol{F}^{-1}$ have the following component form

$$ {}_{(\ell)}\boldsymbol{F} = \delta^i_J \, {}_{(\ell)}\boldsymbol{g}_i \otimes {}_{(\ell)}\boldsymbol{G}^J, \qquad {}_{(\ell)}\boldsymbol{F}^{-1} = \delta^I_j \, {}_{(\ell)}\boldsymbol{G}_I \otimes {}_{(\ell)}\boldsymbol{g}^j. \tag{39} $$

The Jacobian determinants related to the deformation maps ${}_{(\ell)}\boldsymbol{\Phi}_0$, ${}_{(\ell)}\boldsymbol{\Phi}_t$, and ${}_{(\ell)}\boldsymbol{\chi}_t$ are given below

$$ {}_{(\ell)}j_0 := \det\left[\mathrm{GRAD}_{(\ell)}\boldsymbol{\Phi}_0\right] = \left({}_{(\ell)}\boldsymbol{G}_1 \times {}_{(\ell)}\boldsymbol{G}_2\right)\cdot{}_{(\ell)}\boldsymbol{G}_3, \tag{40} $$

$$ {}_{(\ell)}j_t := \det\left[\mathrm{GRAD}_{(\ell)}\boldsymbol{\Phi}_t\right] = \left({}_{(\ell)}\boldsymbol{g}_1 \times {}_{(\ell)}\boldsymbol{g}_2\right)\cdot{}_{(\ell)}\boldsymbol{g}_3, \tag{41} $$

$$ {}_{(\ell)}J_t := \det\left[\mathrm{GRAD}_{(\ell)}\boldsymbol{\chi}_t\right] = \frac{{}_{(\ell)}j_t}{{}_{(\ell)}j_0}, \tag{42} $$

where Eq. (42) is obtained from Eqs. (40) and (41) and from Eq. (37).

Now, we record the results related to the time derivative. From Eqs. (28), (25) and (27), we have

$$ \dot{\boldsymbol{u}} = \dot{\bar{\boldsymbol{u}}} = {}_0\dot{\boldsymbol{\varphi}}, \qquad \ddot{\boldsymbol{u}} = \ddot{\bar{\boldsymbol{u}}} = {}_{(0)}\ddot{\boldsymbol{\varphi}}. \tag{43} $$

Let us write the combined space and kth time derivative of, say, ${}_{(0)}\boldsymbol{\varphi}$ as

$$ {}_{(0)}\overset{\{k\}}{\boldsymbol{\varphi}}_{,\alpha} := \frac{\partial}{\partial \xi^\alpha} \frac{\partial^k}{\partial t^k} {}_{(0)}\boldsymbol{\varphi}, \tag{44} $$

for $\alpha = 0, 1, 2$ and $k = 0, 1, 2$, where the case with $\alpha = 0$ or $k = 0$ means taking no derivative. Then, using Eqs. (44) and (43) in Eqs. (18) and (19), we obtain

$$ {}_{(s\ell)}\overset{\{k\}}{\boldsymbol{\varphi}}_{,\alpha} := \underbrace{{}_{(s[\ell-1])}\overset{\{k\}}{\boldsymbol{\varphi}}{}^{+s}_{,\alpha}}_{[1]} + \underbrace{s\,{}_{(s\ell)}h^{-s}\,{}_{(s\ell)}\overset{\{k\}}{\boldsymbol{t}}_{,\alpha}}_{[2]}, \quad \forall \xi^{\underline{\alpha}} \in \mathscr{A} \tag{45} $$

where

$$ {}_{(s[\ell-1])}\overset{\{k\}}{\boldsymbol{\varphi}}{}^{+s}_{,\alpha} := \underbrace{\overset{\{k\}}{\boldsymbol{u}}_{,\alpha}}_{[1]} + s\left[\underbrace{-{}_{(0)}h^{-s}\,{}_{(0)}\overset{\{k\}}{\boldsymbol{t}}_{,\alpha}}_{[2]} + \underbrace{\sum_{i=0}^{(s-\ell)} {}_{(i)}\underset{(i)}{H}\,\overset{\{k\}}{\boldsymbol{t}}_{,\alpha}}_{[3]}\right], \tag{46} $$

for $k = 0, 1, 2$ and $\alpha = 0, 1, 2$.

Remark 2.6. *To make it easy to understand relations* (45) and (46), *we give below some particular examples. First, for* $(\ell) = 0$ *(reference layer),* $k = 1$, *and* $\alpha = 0$, *we obtain from* Eqs. (45) *and* (46) *using Remark 2.2 (the summation term does not exist for* $\ell = 0$)

$$ {}_{(0)}\dot{\boldsymbol{\varphi}} = \dot{\boldsymbol{u}}. \tag{47} $$

Next, for $\ell = 1$, $k = 1$, *and* $\alpha = 0$, *we obtain from* Eqs. (45) *and* (46) *using* Eq. (4)

$$ {}_{(1)}\dot{\boldsymbol{\varphi}} = \dot{\boldsymbol{u}} + {}_{(0)}h^{+}\,{}_{(0)}\dot{\boldsymbol{t}} + {}_{(1)}h^{-}\,{}_{(1)}\dot{\boldsymbol{t}}. \tag{48} $$

The time rate of the director ${}_{(\ell)}\boldsymbol{t}$ is related to the angular velocity vector ${}_{(\ell)}\boldsymbol{\omega}$, which is perpendicular to ${}_{(\ell)}\boldsymbol{t}$, as follows

$$ {}_{(\ell)}\dot{\boldsymbol{t}} = {}_{(\ell)}\boldsymbol{\omega} \times {}_{(\ell)}\boldsymbol{t}, \quad {}_{(\ell)}\ddot{\boldsymbol{t}} = {}_{(\ell)}\dot{\boldsymbol{\omega}} \times {}_{(\ell)}\boldsymbol{t} + {}_{(\ell)}\boldsymbol{\omega} \times ({}_{(\ell)}\boldsymbol{\omega} \times {}_{(\ell)}\boldsymbol{t}), \quad {}_{(\ell)}\boldsymbol{\omega}\cdot{}_{(\ell)}\boldsymbol{t} = 0. \tag{49} $$

Based on Eq. $(39)_1$, the time derivative of the layer deformation gradient ${}_{(\ell)}\boldsymbol{F}$ takes the form

$$ {}_{(\ell)}\dot{\boldsymbol{F}} = \delta^i_J \, {}_{(\ell)}\dot{\boldsymbol{g}}_i \otimes {}_{(\ell)}\boldsymbol{G}^J, \tag{50} $$

where the computation of ${}_{(\ell)}\dot{\boldsymbol{g}}_i$ is based on Eqs. (30), (31), (45) and (46).

3. Equations of motion

We derive the equations of motion for multilayer geometrically-exact structures based on the principle of virtual power, as pioneered for sandwich beams in Vu-Quoc and Ebcioğlu (1995). Here, we generalize the equations for geometrically-exact sandwich shells derived in Vu-Quoc et al. (1997b) to the case of multilayer shells with unlimited number of layers and with arbitrary reference layer (0). For computational formulation, the weak form is readily obtained from the balance of power.

3.1. Power of contact forces/couples

The shell resultant stresses and resultant couples, and their respective conjugate strain measures can be obtained by reducing the expression of the stress power from an integration over a 3D domain of the shell to an integration over the 2D domain of its material surface. Let $\boldsymbol{P}$ be the first Piola–Kirchhoff stress tensor, then the power of contact forces (or stress power) in the shell expressed in the initial configuration $\mathscr{B}_0$ is

$$ \mathscr{P}_c := \int_{\mathscr{B}_0} \boldsymbol{P} \cdot \cdot \dot{\boldsymbol{F}} \, \mathrm{d}\mathscr{B}_0, \tag{51} $$

where $\mathrm{d}\mathscr{B}_0$ is the infinitesimal volume in the initial configuration $\mathscr{B}_0$.[6]

The first Piola–Kirchhoff stress tensor ${}_{(\ell)}\boldsymbol{P}$ in layer (ℓ) can be written as (see, e.g., Malvern, 1969, p. 222)

$$ {}_{(\ell)}\boldsymbol{P} = {}_{(\ell)}J_t \, {}_{(\ell)}\boldsymbol{F}^{-1} \cdot {}_{(\ell)}\boldsymbol{\sigma}, \tag{52} $$

where ${}_{(\ell)}J_t$ is defined in Eq. (42), ${}_{(\ell)}\boldsymbol{F}^{-1}$ in Eq. (39), and ${}_{(\ell)}\boldsymbol{\sigma}$ is the Cauchy stress tensor in layer (ℓ). Using expression (39) for ${}_{(\ell)}\boldsymbol{F}^{-1}$, the first Piola–Kirchhoff stress tensor ${}_{(\ell)}\boldsymbol{P} = {}_{(\ell)}P^{Ij} \, {}_{(\ell)}\boldsymbol{G}_I \otimes {}_{(\ell)}\boldsymbol{g}_j$ can be expressed as

$$ {}_{(\ell)}\boldsymbol{P} = {}_{(\ell)}J_t \left[\delta^I_i \, {}_{(\ell)}\boldsymbol{G}_I \otimes {}_{(\ell)}\boldsymbol{g}^i \right] \cdot \left[{}_{(\ell)}\sigma^{jk} \, {}_{(\ell)}\boldsymbol{g}_j \otimes {}_{(\ell)}\boldsymbol{g}_k \right] = {}_{(\ell)}J_t \delta^I_i \, {}_{(\ell)}\sigma^{ij} \, {}_{(\ell)}\boldsymbol{G}_I \otimes {}_{(\ell)}\boldsymbol{g}_j. \tag{53} $$

With ${}_{(\ell)}\dot{\boldsymbol{F}}$ given in Eq. (50), we obtain the following stress power per unit reference volume for layer (ℓ)

$$ {}_{(\ell)}\boldsymbol{P} \cdot \cdot {}_{(\ell)}\dot{\boldsymbol{F}} := {}_{(\ell)}J_t \left[\left(\delta^I_i \, {}_{(\ell)}\boldsymbol{G}_I \otimes {}_{(\ell)}\boldsymbol{g}^i \right) \cdot {}_{(\ell)}\boldsymbol{\sigma} \right] \cdot\cdot \left[\delta^k_J \, {}_{(\ell)}\dot{\boldsymbol{g}}_k \otimes {}_{(\ell)}\boldsymbol{G}^J \right] $$

[6] The horizontal double contraction "$\cdot\cdot$" in Eq. (51) is defined as follows. Let $\{\mathbf{e}_a\}$ be a set of fixed cartesian spatial basis vectors. With $\boldsymbol{P}$ and $\dot{\boldsymbol{F}}$ expressed in the bases $\{\mathbf{e}_a\}$ and $\{\mathbf{E}_A\}$ as $\boldsymbol{P} = P^{Aa}\mathbf{E}_A \otimes \mathbf{e}_a$ and $\dot{\boldsymbol{F}} = \dot{F}^a_A \mathbf{e}_a \otimes \mathbf{E}^A$, we have $\boldsymbol{P} \cdot \cdot \dot{\boldsymbol{F}} = P^{Aa}\dot{F}^a_A$ (see Malvern, 1969, p. 35). Note that the convention for the matrix representation of these tensors are as follows: In P^{Aa}, A is the row index, and a the column index, whereas in $\dot{F}^a_A$, a is the row index, and A the column index.

$$= {}_{(\ell)}J_t \delta_i^I \delta_J^k \left({}_{(\ell)}\boldsymbol{G}_I \cdot {}_{(\ell)}\boldsymbol{G}^J \right) {}_{(\ell)}\boldsymbol{g}^i \cdot {}_{(\ell)}\boldsymbol{\sigma} \cdot {}_{(\ell)}\dot{\boldsymbol{g}}_k$$

$$= {}_{(\ell)}J_{t\,(\ell)}\boldsymbol{g}^i \cdot {}_{(\ell)}\boldsymbol{\sigma} \cdot {}_{(\ell)}\dot{\boldsymbol{g}}_i, \tag{54}$$

in which we had used ${}_{(\ell)}\boldsymbol{G}_I \cdot {}_{(\ell)}\boldsymbol{G}^J = \delta_I^J$, according to Eq. $(36)_2$. The stress power for the multilayer shell can now be obtained as

$$\mathscr{P}_c := \int_{B_0} \boldsymbol{P} \cdot \cdot \dot{\boldsymbol{F}} \, \mathrm{d}\mathscr{B}_0 = \sum_{\ell \in \underline{\mathscr{N}}} \int_{(\ell)\mathscr{B}_0} {}_{(\ell)}\boldsymbol{P} \cdot \cdot {}_{(\ell)}\dot{\boldsymbol{F}} \, \mathrm{d}_{(\ell)}\mathscr{B}_0$$

$$= \sum_{\ell \in \underline{\mathscr{N}}} \int_{(\ell)\mathscr{B}_0} {}_{(\ell)}J_{t\,(\ell)}\boldsymbol{g}^i \cdot {}_{(\ell)}\boldsymbol{\sigma} \cdot {}_{(\ell)}\dot{\boldsymbol{g}}_i \, \mathrm{d}_{(\ell)}\mathscr{B}_0, \tag{55}$$

where ${}_{(\ell)}\mathscr{B}_0$ designates the initial configuration for layer (ℓ), and $\mathrm{d}_{(\ell)}\mathscr{B}_0$ the infinitesimal initial volume in layer (ℓ).[7] Using the definition of ${}_{(\ell)}J_t$ in Eq. (42) and of ${}_{(\ell)}j_t$ in Eq. (41), and by the conservation of mass, we have

$${}_{(\ell)}J_t \, \mathrm{d}_{(\ell)}\mathscr{B}_0 = \mathrm{d}_{(\ell)}\mathscr{B}_t = {}_{(\ell)}j_t \, \mathrm{d}_{(\ell)}\mathscr{B}, \tag{56}$$

where $\mathrm{d}_{(\ell)}\mathscr{B}_t$ and $\mathrm{d}_{(\ell)}\mathscr{B} := \mathrm{d}\mathscr{A} \, \mathrm{d}\xi^3$ (with $\mathrm{d}\mathscr{A} = \mathrm{d}\xi^1 \, \mathrm{d}\xi^2$) are the infinitesimal volumes of layer (ℓ) in $\mathscr{B}_t$ and in $\mathscr{B}$, respectively. Using the definition $(8)_1$ of the material domain ${}_{(\ell)}\mathscr{B}$, the definitions (30) and (31) of the basis vector ${}_{(\ell)}\boldsymbol{g}_i$, and expression (56) above, the stress power (55) is now expressed in the material domain as follows[8]

$$\mathscr{P}_c := \sum_{s\ell \in \underline{\mathscr{N}}} \int_{\mathscr{A}} \int_{(s\ell)\mathscr{H}} {}_{(s\ell)}j_t \left\{ \left({}_{(s\ell)}\boldsymbol{g}^\alpha \cdot {}_{(s\ell)}\boldsymbol{\sigma} \right) \cdot \left[{}_{(s\ell)}\dot{\boldsymbol{\varphi}}_{,\alpha} + \left(\xi^3 - s_{(s\ell)}Z \right) {}_{(s\ell)}\dot{\boldsymbol{t}}_{,\alpha} \right] + {}_{(s\ell)}\boldsymbol{g}^3 \cdot {}_{(s\ell)}\boldsymbol{\sigma} \cdot {}_{(s\ell)}\dot{\boldsymbol{t}} \right\} \mathrm{d}\mathscr{A} \, \mathrm{d}\xi^3. \tag{57}$$

Using the following definition of the *weighted* resultant force ${}_{(\ell)}\boldsymbol{n}^\alpha$, the *weighted* resultant couple ${}_{(\ell)}\tilde{\boldsymbol{m}}^\alpha$, and the *weighted* resultant director couple ${}_{(\ell)}\ell$,

$${}_{(\ell)}\boldsymbol{n}^\alpha := \int_{(\ell)\mathscr{H}} {}_{(\ell)}j_{t\,(\ell)}\boldsymbol{g}^\alpha \cdot {}_{(\ell)}\boldsymbol{\sigma} \, \mathrm{d}\xi^3, \quad \forall \ell \in \underline{\mathscr{N}}, \tag{58}$$

$${}_{(s\ell)}\tilde{\boldsymbol{m}}^\alpha := \int_{(\ell)\mathscr{H}} {}_{(s\ell)}j_t \left(\xi^3 - s_{(s\ell)}Z \right) {}_{(s\ell)}\boldsymbol{g}^\alpha \cdot {}_{(s\ell)}\boldsymbol{\sigma} \, \mathrm{d}\xi^3, \quad \forall s\ell \in \underline{\mathscr{N}}, \tag{59}$$

$${}_{(\ell)}\ell := \int_{(\ell)\mathscr{H}} {}_{(\ell)}j_{t\,(\ell)}\boldsymbol{g}^3 \cdot {}_{(\ell)}\boldsymbol{\sigma} \, \mathrm{d}\xi^3, \quad \forall \ell \in \underline{\mathscr{N}}, \tag{60}$$

the stress power (57) can be simplified as

[7] See Eq. $(8)_2$ for the definition of the material configuration ${}_{(\ell)}\mathscr{B}$ of layer (ℓ).

[8] There are some misprints in Eq. (38), p. 2525, in Vu-Quoc et al. (1997b), which is the restriction of Eq. (57) to the case of sandwich shells: ${}_{(\ell)}\boldsymbol{t}$ should be ${}_{(\ell)}\boldsymbol{g}^3$.

$$\mathscr{P}_c = \sum_{\ell \in \underline{\mathscr{N}}} \int_{\mathscr{A}} \left[\underbrace{{}_{(\ell)}\boldsymbol{n}^\alpha \cdot {}_{(\ell)}\dot{\boldsymbol{\varphi}}_{,\alpha}}_{[1]} + \underbrace{{}_{(\ell)}\tilde{\boldsymbol{m}}^\alpha \cdot {}_{(\ell)}\dot{\boldsymbol{t}}_{,\alpha}}_{[2]} + \underbrace{{}_{(\ell)}\boldsymbol{\ell} \cdot {}_{(\ell)}\dot{\boldsymbol{t}}}_{[3]} \right] \mathrm{d}\mathscr{A}. \tag{61}$$

Remark 3.1. *The integration in* Eq. (61) *is carried out on the material configuration* $\mathscr{B}$, *and not on the current configuration* $\mathscr{B}_t$.

Remark 3.2. *Note that our definition of the resultant tensors* ${}_{(\ell)}\boldsymbol{n}^\alpha$, ${}_{(\ell)}\tilde{\boldsymbol{m}}^\alpha$, ${}_{(\ell)}\boldsymbol{\ell}$ *shown above corresponds to the* weighted resultant tensors[9] *known as the "weighted surface tensors" in* Green and Zerna (1968, p. 375), *and differ from the resultant tensors* $\boldsymbol{n}^\alpha$, $\tilde{\boldsymbol{m}}^\alpha$, $\boldsymbol{\ell}$ *defined in* Simo and Fox (1989, p. 282) *by a factor* $1/\bar{j}$, *where*

$${}_{(\ell)}\bar{j}_t := {}_{(\ell)}j_t \Big|_{\xi^3 = {}_{(\ell)}Z}, \tag{62}$$

i.e., the value of the Jacobian ${}_{(\ell)}j_t$ *(defined in* Eq. (41)) *at the neutral surface of layer* (ℓ) *(defined in* Eq. (3)). *In fact, among the above three weighted resultant tensors, only our resultant* ${}_{(\ell)}\boldsymbol{n}^\alpha$ *has the following equivalent in* Green and Zerna (1968, p. 375, Eq. $(10.2.8)_1$)

$$\mathbf{N}_\alpha = \int_{-(1/2)t}^{(1/2)t} \mathbf{T}_\alpha \, \mathrm{d}\theta^3,$$

where $\mathbf{T}_\alpha$ *is exactly the same as* $[{}_{(\ell)}j_{t\,(\ell)}\boldsymbol{g}^\alpha \cdot {}_{(\ell)}\boldsymbol{\sigma}]$ *in our notation. On the other hand*, Green and Zerna (1968) *did not have the equivalent of the other two weighted resultant tensors, i.e.,* ${}_{(\ell)}\tilde{\boldsymbol{m}}^\alpha$ *and* ${}_{(\ell)}\boldsymbol{\ell}$. *The definition of* $\boldsymbol{n}^\alpha$ *in* Simo and Fox (1989) *corresponds to the quantity* $\mathbf{N}_\alpha/\sqrt{a}$ *used in the definition of the "stress resultant"* **n** *in* Green and Zerna (1968, p. 377, Eq. $(10.2.18)_1$), *as reproduced below*

$$\mathbf{n} = \frac{v_\alpha \mathbf{N}_\alpha}{\sqrt{a}},$$

where $\sqrt{a}$ *is the same as* $\bar{j}$. *See also* Remark 4.1 *on the definitions of the resultant moments. An advantage of our definition is that it simplifies significantly the equations of motion for multilayer shells to be presented in* Section 3.4. *In particular,* ${}_{(\ell)}\tilde{\boldsymbol{m}}^\alpha$ *is the* weighted resultant couple, *and not the* true resultant couple ${}_{(\ell)}\boldsymbol{m}^\alpha$ *to be introduced later in* Eq. $(133)_1$. *For more details on the dimensions of these tensors, we refer the readers to* Vu-Quoc et al. (1997b). *Note that even though integrated in the material configuration* $\mathscr{B}$, *the weighted resultant tensors are actually* spatial *tensors defined on the current configuration* $\mathscr{B}_t$.

Remark 3.3. *It should be noted that for* $i \in \{1, 2, 3\}$ *fixed, the normal* ${}_{(\ell)}\boldsymbol{g}^i$ *is in general different for different layer* (ℓ), *i.e.,* ${}_{(p)}\boldsymbol{g}^i \neq {}_{(q)}\boldsymbol{g}^i$, *for* $p \neq q$. *It follows that the same remark applies to all weighted resultant tensors* ${}_{(\ell)}\boldsymbol{n}^\alpha$, ${}_{(\ell)}\tilde{\boldsymbol{m}}^\alpha$, ${}_{(\ell)}\boldsymbol{\ell}$.

Remark 3.4. *The power* $\mathscr{P}_c$ *in* Eq. (61) *of contact forces/couples is expressed with respect to the weighted resultant tensors explained in* Remark 3.2. *For computation, we need to account for the constitutive restriction in* $\mathscr{P}_c$, *which can then be expressed in terms of physical quantities such as the membrane forces, moments, and shear forces, which are conjugate to the rate of membrane strains, curvatures, and shear*

[9] Weighted tensors are also called relative tensors; see, e.g., McConnell (1957).

strain, as will be shown later in Section 4.3.

Next, to derive the equations of motion in terms of the weighted resultant tensors, called the *weighted resultant equations of motion*, we express the power $\mathscr{P}_c$ in terms of the time rate of the principal unknown kinematic quantities $\boldsymbol{u}$ and ${}_{(\ell)}\boldsymbol{t}$, for $\ell \in \underline{\mathscr{N}}$, which are discussed in Section 2.2. The methodology employed here follows closely that are presented in Vu-Quoc and Ebcioğlu (1996) for multilayer beams.

First, we consider the terms in Eq. (61) with factors $\dot{\boldsymbol{u}}_{,\alpha}$; these terms come from Part [1] of Eq. (61), due to the expression of ${}_{(s\ell)}\dot{\boldsymbol{\varphi}}_{,\alpha}$ as given in Part [1] of Eqs. (45) and (46). These terms constitute the linear momentum part of the stress power $\mathscr{P}_c$, and is defined as follows

$$\mathscr{L} := \int_{\mathscr{A}} \hat{\boldsymbol{n}}^{\alpha} \cdot \dot{\boldsymbol{u}}_{,\alpha} \,\mathrm{d}\mathscr{A}, \quad \hat{\boldsymbol{n}}^{\alpha} := \sum_{\ell \in \underline{\mathscr{N}}} {}_{(\ell)}\boldsymbol{n}^{\alpha}, \tag{63}$$

where $\hat{\boldsymbol{n}}^{\alpha}$ represents the *total* resultant contact force. So now, to have the above expression for $\mathscr{L}$ in terms of the time rate $\dot{\boldsymbol{u}}$, we integrate by parts to obtain

$$\mathscr{L} = \oint_{\partial\mathscr{A}} \hat{\boldsymbol{n}}^{\alpha} \cdot \dot{\boldsymbol{u}}_{(0)}\nu_{\alpha} \,\mathrm{d}(\partial\mathscr{A}) - \int_{\mathscr{A}} \hat{\boldsymbol{n}}^{\alpha}_{,\alpha} \cdot \dot{\boldsymbol{u}} \,\mathrm{d}\mathscr{A}, \tag{64}$$

where we have made use of Eq. (11).

Next, we consider the *weighted* angular momentum terms for $\mathscr{P}_c$ for the reference layer (0) in expression (61). To this end, we consider the terms with factors ${}_{(0)}\dot{\boldsymbol{t}}$ and ${}_{(0)}\dot{\boldsymbol{t}}_{,\alpha}$ in Eq. (61):[10]

$$ {}_{(0)}\aleph := \int_{\mathscr{A}} \Bigg[\Bigg(\underbrace{\mathfrak{M}^{\alpha}}_{[1]} + \underbrace{{}_{(0)}\tilde{\boldsymbol{m}}^{\alpha}}_{[2]} \Bigg) \cdot {}_{(0)}\dot{\boldsymbol{t}}_{,\alpha} + \underbrace{{}_{(0)}\ell \cdot {}_{(0)}\dot{\boldsymbol{t}}}_{[3]} \Bigg] \mathrm{d}\mathscr{A}. \tag{65}$$

where

$$\mathfrak{M}^{\alpha} := {}_{(0)}h^{+} \sum_{j=1}^{\hat{\mathscr{N}}} {}_{(j)}\boldsymbol{n}^{\alpha} - {}_{(0)}h^{-} \sum_{j=-1}^{-\mathscr{N}} {}_{(j)}\boldsymbol{n}^{\alpha} = \sum_{s\ell \in \underline{\mathscr{N}} \setminus \{0\}} s_{(0)}h^{+s}{}_{(s\ell)}\boldsymbol{n}^{\alpha} \tag{66}$$

is the total moment of the resultant forces ${}_{(\ell)}\boldsymbol{n}^{\alpha}$ of all top layers with respect to the top surface of the reference layer (0), and of all bottom layers with respect to the bottom surface of the reference layer (0). In Eq. (65), Parts [1], [2], [3] come from Parts [1], [2], [3] of Eq. (61), respectively. To be more specific, Part [1] of Eq. (65) comes from Part [1] of Eqs. (61) and (45) with $k = 1$, and Parts [2] and [3] of Eq. (46), and the definition of ${}_{(i)}H$ in Eq. (4). Note that in the definition of $\mathfrak{M}^{\alpha}$ in Eq. (66), the resultant ${}_{(0)}\boldsymbol{n}^{\alpha}$ of layer (0) is absent in the summation since when $\ell = 0$ there is a cancellation of Part [2] in Eq. (45) ($\ell = 0$) and Part [2] in Eq. (46), keeping in mind that Part [3] is nonexistent for $\ell = 0$ due to Remark 2.2.

To have ${}_{(0)}\dot{\boldsymbol{t}}$ as the common factor in the expression for ${}_{(0)}\aleph$, we integrate by parts the first term of Eq. (65), i.e., Parts [1] and [2], to obtain

$${}_{(0)}\aleph = \oint_{\partial\mathscr{A}} \left(\mathfrak{M}^{\alpha} + {}_{(0)}\tilde{\boldsymbol{m}}^{\alpha}\right) \cdot {}_{(0)}\dot{\boldsymbol{t}}_{(0)}\nu_{\alpha} \,\mathrm{d}(\partial\mathscr{A}) - \int_{\mathscr{A}} \Big[\left(\mathfrak{M}^{\alpha} + {}_{(0)}\tilde{\boldsymbol{m}}^{\alpha}\right)_{,\alpha} - {}_{(0)}\ell \Big] \cdot {}_{(0)}\dot{\boldsymbol{t}} \,\mathrm{d}\mathscr{A}. \tag{67}$$

[10] The Hebrew character 'aleph ℵ is the equivalent form of the Latin character A, which is mnemonic for "angular".

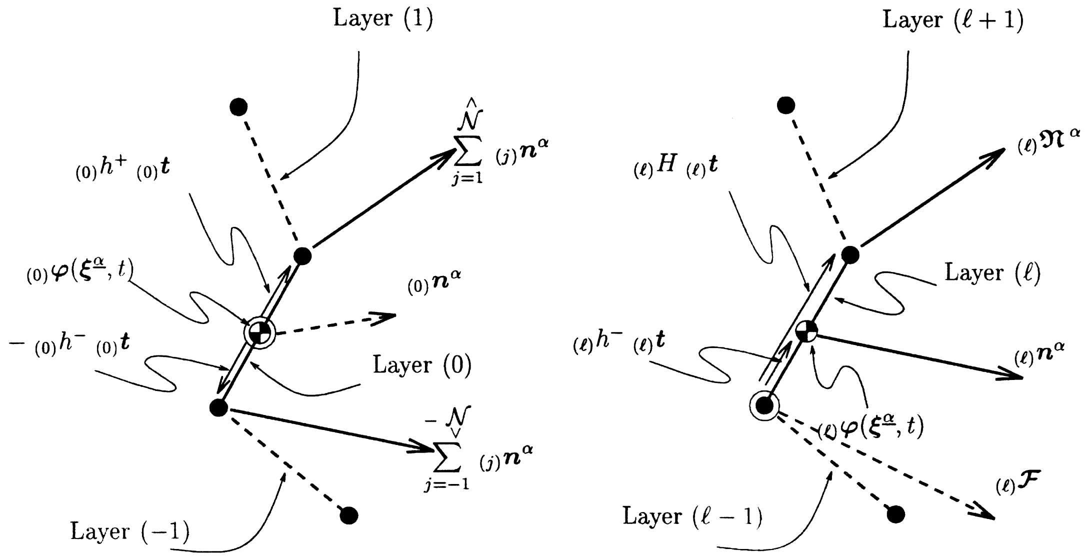

Fig. 6. *Meaning of coupling terms*: For layer (0) (left figure), take moment about layer center. For upper layer (ℓ) (right figure), take moment about bottom hinge.

We introduce the following functions ${}_{(0)}\overline{\mathscr{M}}^{\alpha}: \partial\mathscr{A} \to \mathbb{R}$ and ${}_{(0)}\mathscr{M}: \mathscr{A} \to \mathbb{R}$ to write Eq. (67) in a more compact form as follows

$$ {}_{(0)}\aleph = \oint_{\partial\mathscr{A}} {}_{(0)}\overline{\mathscr{M}}^{\alpha} \cdot {}_{(0)}\dot{\boldsymbol{t}}\, {}_{(0)}\nu_{\alpha} \, \mathrm{d}(\partial\mathscr{A}) - \int_{\mathscr{A}} {}_{(0)}\mathscr{M} \cdot {}_{(0)}\dot{\boldsymbol{t}} \, \mathrm{d}\mathscr{A}. \tag{68} $$

Now we consider the terms in the mechanical power (61) that have ${}_{(s\ell)}\dot{\boldsymbol{t}}$ and ${}_{(s\ell)}\dot{\boldsymbol{t}}_{,\alpha}$, $\forall s\ell \in \underline{\mathscr{N}} \backslash \{0\}$, as factors. For convenience in presenting the equations, we introduce the following definitions. Let us combine the top-layer number and the bottom-layer number into the compact expression $s\mathbb{N}$ defined as

$$ s\mathbb{N} := \begin{cases} \hat{\mathscr{N}}, & \text{for } s = +1, \\ -\check{\mathscr{N}}, & \text{for } s = -1. \end{cases} \tag{69} $$

Consider layer ($s\ell$); let ${}_{(s\ell)}\mathfrak{N}^{\alpha}$ be the resultant force, acting on the facet with normal ${}_{(s\ell)}\boldsymbol{g}^{\alpha}$ of layer ($s\ell$), due to the contribution from layer ($s(\ell+1)$) to layer ($s\mathbb{N}$) (see also Fig. 6 for a geometric representation of ${}_{(s\ell)}\mathfrak{N}^{\alpha}$):[11]

$$ {}_{(s\ell)}\mathfrak{N}^{\alpha} := \sum_{i=s(\ell+1)}^{s\mathbb{N}} {}_{(i)}\boldsymbol{n}^{\alpha}, \quad \forall s\ell \in \underline{\mathscr{N}} \backslash \{0\}. \tag{70} $$

From Eq. (61), we obtain the following weighted angular momentum for layer ($s\ell$), with $s\ell \in \underline{\mathscr{N}} \backslash \{0\}$, having terms with factors ${}_{(s\ell)}\dot{\boldsymbol{t}}$ and ${}_{(s\ell)}\dot{\boldsymbol{t}}_{,\alpha}$

[11] That is, in Eq. (70), if $s = +1$ (i.e., upper layer), then $i = (\ell+1), \ldots, \hat{\mathscr{N}}$; if $s = -1$ (i.e., lower layer), then $i = -(\ell+1), \ldots, -\check{\mathscr{N}}$.

$$ {}_{(s\ell)}\aleph := \int_{\mathscr{A}} \left\{ s \left[\underbrace{{}_{(s\ell)}h^{-s}{}_{(s\ell)}\boldsymbol{n}^{\alpha}}_{[1]} + \underbrace{{}_{(s\ell)}H_{(s\ell)}\mathfrak{N}^{\alpha}}_{[2]} \right] \cdot {}_{(s\ell)}\dot{\boldsymbol{t}}_{,\alpha} + \underbrace{{}_{(s\ell)}\tilde{\boldsymbol{m}}^{\alpha} \cdot {}_{(s\ell)}\dot{\boldsymbol{t}}_{,\alpha}}_{[3]} + \underbrace{{}_{(s\ell)}\boldsymbol{\ell} \cdot {}_{(s\ell)}\dot{\boldsymbol{t}}}_{[4]} \right\} \mathrm{d}\mathscr{A}, \tag{71} $$

where Part [1] of Eq. (71) comes from Part [1] of Eq. (61) and Part [2] of Eq. (45) with $k = 1$; Part [2] of Eq. (71) comes from Part [1] of Eq. (61) and Part [3] of Eq. (46) (see Remark 3.5); Parts [3] and [4] of Eq. (71) come from Parts [2] and [3] of Eq. (61), respectively.

Remark 3.5. *Note that Part* [2] *in* Eq. (71) *is made up of the contributions of layers* $\{(s[(\ell)+1]),\ldots,s\mathbb{N}\}$ *to layer* $(s\ell)$ *as shown in the definition of* ${}_{(s\ell)}\mathfrak{N}^{\alpha}$ *in* Eq. (70). *To understand* Eq. (70), *consider the special case where* $s\ell = \hat{\ell} > 0$ *in both* Eqs. (71) and (70). *Recall that to obtain Part* [2] *in* Eq. (71), *we need to gather all terms in* Eq. (61) *having* ${}_{(\hat{\ell})}\dot{\boldsymbol{t}}_{,\alpha}$ *as a factor. To this end, consider the summation with index* ℓ *in* Eq. (61). *When* $\ell \leq \hat{\ell}$, *there is no term in Part* [3] *of* (46) $(k = 1)$ *having* ${}_{(\hat{\ell})}\dot{\boldsymbol{t}}_{,\alpha}$ *as a factor, since the upper limit of the summation is strictly less than* $\hat{\ell}$. *For* $\ell > \hat{\ell}$, *i.e.*, $\ell = (\hat{\ell}+1),\ldots,\hat{\mathscr{N}}$, *there is the term* ${}_{(\hat{\ell})}H_{(\hat{\ell})}\dot{\boldsymbol{t}}_{,\alpha}$ *in Part* [3] *of* Eq. (46) $(k = 1)$. *Thus picking out all terms having* ${}_{(\hat{\ell})}\dot{\boldsymbol{t}}_{,\alpha}$ *as a factor in Part* [1] *of* Eq. (61), *i.e.*, $\sum_{(\ell)\in\underline{\mathscr{N}}}{}_{(\ell)}\boldsymbol{n}^{\alpha}\cdot{}_{(\ell)}\dot{\boldsymbol{\varphi}}_{,\alpha}$, *we have*

$$ \left(\sum_{\ell=\hat{\ell}+1}^{\hat{\mathscr{N}}} {}_{(\ell)}\boldsymbol{n}^{\alpha} \right) \cdot {}_{(\hat{\ell})}\dot{\boldsymbol{t}}_{,\alpha} = {}_{(\hat{\ell})}\mathfrak{N}^{\alpha} \cdot {}_{(\hat{\ell})}\dot{\boldsymbol{t}}_{,\alpha}. \tag{72} $$

Generalizing the above result to both top and bottom layers, we obtain Eq. (70) *and Part* [2] *of* Eq. (71).

To have ${}_{(s\ell)}\dot{\boldsymbol{t}}$ as the common factor, we integrate by parts the first three terms in the integrand of Eq. (71) to obtain

$$ {}_{(s\ell)}\aleph = \oint_{\partial\mathscr{A}} \left[s\left({}_{(s\ell)}h^{-s}{}_{(s\ell)}\boldsymbol{n}^{\alpha} + {}_{(s\ell)}H_{(s\ell)}\mathfrak{N}^{\alpha} \right) + {}_{(s\ell)}\tilde{\boldsymbol{m}}^{\alpha} \right] \cdot {}_{(s\ell)}\dot{\boldsymbol{t}}_{(0)}\nu_{\alpha}\, \mathrm{d}(\partial\mathscr{A}) - \int_{\mathscr{A}} \Big\{ \Big[s\big({}_{(s\ell)}h^{-s}{}_{(s\ell)}\boldsymbol{n}^{\alpha} + {}_{(s\ell)}H_{(s\ell)}\mathfrak{N}^{\alpha} \big) + {}_{(s\ell)}\tilde{\boldsymbol{m}}^{\alpha} \Big]_{,\alpha} \cdot {}_{(s\ell)}\dot{\boldsymbol{t}} - {}_{(s\ell)}\boldsymbol{\ell} \cdot {}_{(s\ell)}\dot{\boldsymbol{t}} \Big\} \mathrm{d}\mathscr{A}, \tag{73} $$

where we have made use of Eq. (11). Similarly to Eq. (68), we introduce for layer $(s\ell)$ the following functions ${}_{(s\ell)}\overline{\mathscr{M}}^{\alpha}: \partial\mathscr{A} \to \mathbb{R}$ and ${}_{(s\ell)}\mathscr{M}: \mathscr{A} \to \mathbb{R}$ to write Eq. (73) in a more compact form as follows

$$ {}_{(\ell)}\aleph = \oint_{\partial\mathscr{A}} {}_{(\ell)}\overline{\mathscr{M}}^{\alpha} \cdot {}_{(\ell)}\dot{\boldsymbol{t}}_{(0)}\nu_{\alpha}\, \mathrm{d}(\partial\mathscr{A}) - \int_{\mathscr{A}} {}_{(\ell)}\mathscr{M} \cdot {}_{(\ell)}\dot{\boldsymbol{t}}\, \mathrm{d}\mathscr{A}. \tag{74} $$

The power of the resultant contact forces/couples (or stress power) in Eq. (61) can now be expressed in terms of the newly defined quantities as follows

$$ \mathscr{P}_c = \mathscr{L} + \sum_{(\ell)\in\underline{\mathscr{N}}} {}_{(\ell)}\aleph, \tag{75} $$

where $\mathscr{L}$ is the linear momentum part in $\mathscr{P}_c$ defined in Eq. (64), and ${}_{(\ell)}\aleph$ the angular momentum part for layer (ℓ) in $\mathscr{P}_c$ defined in Eqs. (68) and (74).

3.2. Rate of kinetic energy

For each layer (ℓ), let ${}_{(\ell)}\boldsymbol{v}: {}_{(\ell)}\mathscr{B}_t \times \mathbb{R}_+ \to T_{(\ell)}\mathscr{B}_t$ be the spatial velocity field on the spatial tangent bundle $T_{(\ell)}\mathscr{B}_t$, and ${}_{(\ell)}\rho_t: {}_{(\ell)}\mathscr{B}_t \times \mathbb{R}_+ \to \mathbb{R}$ the mass density defined on ${}_{(\ell)}\mathscr{B}_t$. The kinetic energy of the multilayer shell can be written as

$$\mathscr{K} = \frac{1}{2} \sum_{\ell \in \underline{\mathscr{N}}} \int_{{}_{(\ell)}\mathscr{B}_t} {}_{(\ell)}\rho_t \, {}_{(\ell)}\boldsymbol{v} \cdot {}_{(\ell)}\boldsymbol{v} \, \mathrm{d}\big({}_{(\ell)}\mathscr{B}_t\big). \tag{76}$$

By the Reynolds transport theorem and the conservation of mass (e.g., Malvern, 1969, p. 210) we obtain the material derivative of the kinetic energy in Eq. (76) as

$$\frac{\mathrm{d}\mathscr{K}}{\mathrm{d}t} = \sum_{\ell \in \underline{\mathscr{N}}} \int_{{}_{(\ell)}\mathscr{B}_t} {}_{(\ell)}\rho_t \, {}_{(\ell)}\boldsymbol{a} \cdot {}_{(\ell)}\boldsymbol{v} \, \mathrm{d}\big({}_{(\ell)}\mathscr{B}_t\big), \tag{77}$$

which when expressed in the material configuration $\mathscr{B}$ yields

$$\frac{\mathrm{d}\mathscr{K}}{\mathrm{d}t} = \sum_{(\ell) \in \underline{\mathscr{N}}} \int_{\mathscr{A}} \int_{{}_{(\ell)}\mathscr{H}} {}_{(\ell)}\rho_t \, {}_{(\ell)}\boldsymbol{A} \cdot {}_{(\ell)}\boldsymbol{V} \, {}_{(\ell)}j_t \, \mathrm{d}\mathscr{A} \, \mathrm{d}\xi^3, \tag{78}$$

where ${}_{(\ell)}j_t$ is the Jacobian determinant — defined in Eq. (41) — of the deformation map ${}_{(\ell)}\boldsymbol{\Phi}$ for layer (ℓ) given in Eq. (17), whereas ${}_{(\ell)}\boldsymbol{A}: \mathscr{B} \times \mathbb{R}_+ \rightarrow T\mathscr{B}_t$ and ${}_{(\ell)}\boldsymbol{V}: \mathscr{B} \times \mathbb{R}_+ \rightarrow T\mathscr{B}_t$ are the material acceleration and material velocity expressed in terms of the material configuration $\mathscr{B}$, and is related to ${}_{(\ell)}\boldsymbol{\Phi}$ by (see, e.g., Marsden and Hughes, 1983)

$${}_{(\ell)}\boldsymbol{V}(\xi, t) := \frac{\partial \, {}_{(\ell)}\boldsymbol{\Phi}(\xi, t)}{\partial t} = {}_{(\ell)}\dot{\boldsymbol{\Phi}}(\xi, t), \qquad {}_{(\ell)}\boldsymbol{A}(\xi, t) := \frac{\partial^2 \, {}_{(\ell)}\boldsymbol{\Phi}(\xi, t)}{\partial t^2} = {}_{(\ell)}\ddot{\boldsymbol{\Phi}}(\xi, t). \tag{79}$$

Since we want to integrate Eq. (78) in the thickness coordinate ξ^3 to obtain an expression for the rate of kinetic energy in terms of an integral over the material area $\mathscr{A}$. To this end, we introduce the deformation map ${}_{(\ell)}\boldsymbol{\psi}: {}_{(\ell)}\mathscr{B} \times \mathbb{R}_+ \rightarrow \mathbb{R}^3$, and rewrite Eq. (17) as follows

$${}_{(\ell)}\boldsymbol{\Phi}_t(\xi) = {}_{(\ell)}\boldsymbol{\psi}_t(\xi^{\underline{\alpha}}) + \xi^3 \, {}_{(\ell)}\boldsymbol{t}(\xi^{\underline{\alpha}}), \quad \forall \ell \in \underline{\mathscr{N}}, \tag{80}$$

where from Eqs. (80) and (17), we have[12]

$${}_{(s\ell)}\boldsymbol{\psi} := {}_{(s\ell)}\boldsymbol{\varphi} - s \, {}_{(s\ell)}Z \, {}_{(s\ell)}\boldsymbol{t}. \tag{81}$$

Next, to have an expression for ${}_{(\ell)}\boldsymbol{\varphi}$ in terms of the surface material coordinates $\boldsymbol{\xi}^{\underline{\alpha}}$, the displacement field $\tilde{\boldsymbol{u}}$ defined in Eq. (25), and the layer directors ${}_{(\ell)}\boldsymbol{t}$, we make use of the definition of ${}_{(\ell)}Z$ in Eqs. (5) and (6), and the expression for ${}_{(\ell)}\boldsymbol{\varphi}$ in Eqs. (18) and (19) to obtain

$${}_{(0)}\boldsymbol{\psi} = {}_{(0)}\boldsymbol{\varphi} = \xi^{\alpha} \mathbf{E}_{\alpha} + \tilde{\boldsymbol{u}}(\boldsymbol{\xi}^{\underline{\alpha}}, t), \tag{82}$$

and

$${}_{(s\ell)}\boldsymbol{\psi} = {}_{(0)}\boldsymbol{\varphi} + s \left\{ {}_{(0)}h^{+s} [{}_{(0)}\boldsymbol{t} - {}_{(s\ell)}\boldsymbol{t}] + \sum_{i=s}^{s(\ell-1)} {}_{(i)}H [{}_{(i)}\boldsymbol{t} - {}_{(s\ell)}\boldsymbol{t}] \right\}, \quad \forall s\ell \in \underline{\mathscr{N}} \backslash \{0\}. \tag{83}$$

[12] Recall that ${}_{(\ell)}\boldsymbol{\psi}_t(\boldsymbol{\xi}^{\underline{\alpha}}) \equiv {}_{(\ell)}\boldsymbol{\psi}(\boldsymbol{\xi}^{\underline{\alpha}}, t)$.

Remark 3.6. *Note that* Eq. (82) *is actually a particular case of* Eq. (83) *by* Remark 2.2, *since for* $\ell = 0$, *the term with the summation sign in* Eq. (83) *does not exist, thus making the whole expression in curly brackets in* Eq. (83) *vanishes. For* $s\ell = +1$, *we obtain from* Eq. (83) *the following expression*

$$_{(1)}\boldsymbol{\psi} = {}_{(0)}\boldsymbol{\varphi} + {}_{(0)}h^{+}[{}_{(0)}\boldsymbol{t} - {}_{(1)}\boldsymbol{t}], \tag{84}$$

which is the same as in Vu-Quoc et al. (1997, p. 2528, Eq. (61)).

Next, using Eqs. (79) and (80) in Eq. (78), we obtain the following expression of rate of kinetic energy

$$\frac{\mathrm{d}\mathscr{K}}{\mathrm{d}t} = \sum_{\ell \in \underline{\mathscr{N}}} \int_{\mathscr{A}} \left[{}_{(\ell)}\mathfrak{f} \cdot {}_{(\ell)}\dot{\boldsymbol{\psi}} + {}_{(\ell)}\mathfrak{g} \cdot {}_{(\ell)}\dot{\boldsymbol{t}} \right] \mathrm{d}\mathscr{A}, \tag{85}$$

where $_{(\ell)}\mathfrak{f}$ and $_{(\ell)}\mathfrak{g}$ are the inertia force and inertia couple for layer (ℓ), respectively, and are given by

$$_{(\ell)}\mathfrak{f} := {}_{(\ell)}A^{0}_{\rho(\ell)}\ddot{\boldsymbol{\psi}} + {}_{(\ell)}A^{1}_{\rho(\ell)}\ddot{\boldsymbol{t}}, \qquad {}_{(\ell)}\mathfrak{g} := {}_{(\ell)}A^{1}_{\rho(\ell)}\ddot{\boldsymbol{\psi}} + {}_{(\ell)}A^{2}_{\rho(\ell)}\ddot{\boldsymbol{t}}, \quad \forall (\ell) \in \underline{\mathscr{N}}, \tag{86}$$

with $_{(\ell)}A^{k}_{\rho}$, for $k = 0, 1, 2$, being the kth mass moments defined as follows

$$_{(\ell)}A^{k}_{\rho} := \int_{{}_{(\ell)}\mathscr{H}} {}_{(\ell)}\rho_t \left(\xi^3\right)^k {}_{(\ell)}j_t \, \mathrm{d}\xi^3. \tag{87}$$

Note that, for layer (ℓ), the quantity $_{(\ell)}A^{0}_{\rho}$ is the mass per unit undeformed area, $_{(\ell)}A^{1}_{\rho}$ the mass moment per unit undeformed area, and $_{(\ell)}A^{2}_{\rho}$ the mass moment of inertia per unit undeformed area.

Next, we want to express Eq. (85) in terms of the time rate of the principal kinematic quantities, i.e., $\{\dot{\boldsymbol{u}}, {}_{(\ell)}\dot{\boldsymbol{t}}, \forall \ell \in \underline{\mathscr{N}}\}$. To this end, we need to express $_{(\ell)}\dot{\boldsymbol{\psi}}$ in the first term of Eq. (85) in terms of $\{\dot{\boldsymbol{u}}, {}_{(\ell)}\dot{\boldsymbol{t}}\}$. Such expression is readily obtained from Eqs. (82) and (83), bearing in mind Eq. $(43)_1$. Note that Eq. (83) can be rewritten using the definition of $_{(s\ell)}Y$ in Eq. $(6)_2$ as follows

$$_{(\ell)}\boldsymbol{\psi} = {}_{(0)}\boldsymbol{\varphi} + s\left\{ {}_{(0)}h^{+s}{}_{(0)}\boldsymbol{t} - {}_{(s\ell)}Y_{(s\ell)}\boldsymbol{t} + \sum_{i=s}^{s(\ell-1)} {}_{(i)}H_{(i)}\boldsymbol{t} \right\}. \tag{88}$$

With the following definition of the total resultant inertia force $\hat{\mathfrak{f}}$

$$\hat{\mathfrak{f}} := \sum_{\ell \in \underline{\mathscr{N}}} {}_{(\ell)}\mathfrak{f}, \tag{89}$$

together with Eqs. (82), (88) and (47), we can rewrite Eq. (85) as follows

$$\frac{\mathrm{d}\mathscr{K}}{\mathrm{d}t} = \int_{\mathscr{A}} \left\{ \hat{\mathfrak{f}} \cdot \dot{\boldsymbol{u}} + \left[{}_{(0)}\mathfrak{g} + \sum_{s\ell \in \underline{\mathscr{N}} \backslash \{0\}} s_{(0)}h^{+s}{}_{(s\ell)}\mathfrak{f} \right] \cdot {}_{(0)}\dot{\boldsymbol{t}} + \underbrace{\sum_{s\ell \in \underline{\mathscr{N}} \backslash \{0\}} \left[{}_{(s\ell)}\mathfrak{g} - s_{(s\ell)}Y_{(s\ell)}\mathfrak{f} \right] \cdot {}_{(s\ell)}\dot{\boldsymbol{t}}}_{[1]} \right.$$
$$\left. + \underbrace{\sum_{s\ell \in \underline{\mathscr{N}} \backslash \{0\}} \sum_{i=s}^{s(\ell-1)} s_{(i)}H_{(s\ell)}\mathfrak{f} \cdot {}_{(i)}\dot{\boldsymbol{t}}}_{[2]} \right\} \mathrm{d}\mathscr{A}. \tag{90}$$

Remark 3.7. *In Part* [2] *of* Eq. (90), *due to the upper limit of index i in the second summation, i.e.,* $s(\ell-1)$, *by* Remark 2.2, *the index* $s\ell$ *in the first summation effectively starts with* ± 2, *and not with* ± 1.

We need to rewrite Part [2] of Eq. (90) so to combine with Part [1]. The layer index in ${}_{(i)}\dot{\boldsymbol{t}}$ in Part [2] of Eq. (90) is i, instead of $s\ell$ as in Part [1]. Thus, we want to express Part [2] of Eq. (90) in terms of ${}_{(s\ell)}\dot{\boldsymbol{t}}$, as in Part [1]. To this end, define

$$S_{(i)(s\ell)} := s_{(i)} H_{(s\ell)} \mathfrak{f} \cdot {}_{(i)}\dot{\boldsymbol{t}}. \tag{91}$$

Lemma 3.1. *The following interchanging of summation indices holds*

$$\sum_{s\ell \in \underline{\mathcal{N}} \setminus \{0\}} \sum_{i=s}^{s(\ell-1)} S_{(i)(s\ell)} = \sum_{s\ell \in \underline{\mathcal{N}} \setminus \{0,\ s\mathbb{N}\}} \sum_{i=s(\ell+1)}^{s\mathbb{N}} S_{(s\ell)(i)}. \tag{92}$$

Proof. See Vu-Quoc and Ebcioğlu (1996, Remarks 3.6 and 3.7, p. 400).

With

$$S_{(s\ell)(i)} = s_{(s\ell)} H_{(i)} \mathfrak{f} \cdot {}_{(s\ell)}\dot{\boldsymbol{t}}, \tag{93}$$

we can rewrite Eq. (90) with the aid of Lemma 3.1 as follows

$$\frac{\mathrm{d}\mathscr{K}}{\mathrm{d}t} = \int_{\mathscr{A}} \left[\hat{\mathfrak{f}} \cdot \dot{u}\right] + \sum_{\ell \in \underline{\mathcal{N}}} {}_{(\ell)}\boldsymbol{C} \cdot {}_{(\ell)}\dot{\boldsymbol{t}} \, \mathrm{d}\mathscr{A}, \tag{94}$$

where the total resultant inertia force $\hat{\mathfrak{f}}$ had been defined in Eq. (89), and the inertia couples ${}_{(\ell)}\boldsymbol{C}$ for layer (ℓ) are given below. For the reference layer (0),

$${}_{(0)}\boldsymbol{C} = {}_{(0)}\mathfrak{g} + \sum_{s\ell \in \underline{\mathcal{N}} \setminus \{0\}} s_{(0)} h^{+s}{}_{(s\ell)}\mathfrak{f}. \tag{95}$$

For any other layers, excluding layer (0) and the top and bottom layers ($s\mathbb{N}$), i.e., $\forall s\ell \in \underline{\mathcal{N}} \setminus \{0,\ s\mathbb{N}\}$,

$${}_{(s\ell)}\boldsymbol{C} = {}_{(s\ell)}\mathfrak{g} - s_{(s\ell)} Y_{(s\ell)} \mathfrak{f} + s_{(s\ell)} H \sum_{i=s(\ell+1)}^{s\mathbb{N}} {}_{(i)}\mathfrak{f}. \tag{96}$$

For the top and bottom layers ($s\mathbb{N}$),

$${}_{(s\mathbb{N})}\boldsymbol{C} = {}_{(s\mathbb{N})}\mathfrak{g} - s_{(s\mathbb{N})} Y_{(s\mathbb{N})} \mathfrak{f}. \tag{97}$$

Remark 3.8. *Note that actually* Eq. (97) *is a particular case of* Eq. (96), *since when* $s\ell = s\mathbb{N}$ *in* Eq. (96), *the third summation term does not exist, since the lower bound of the summation index i is* $s(\mathbb{N}+1)$, *which is clearly out of bound.*

Remark 3.9. *From* Eqs. (95)–(97), *we recover the expressions for the inertia couples derived for sandwich shells in* Vu-Quoc et al. (1997b, p. 2530, Eq. (75)).

3.3. Power of assigned forces/couples

Parallel to the power of $\mathscr{P}_c$ contact forces in Eq. (51) for the 3D shell body, expressed in the initial configuration $\mathscr{B}_0$, is the power $\mathscr{P}_a$ of the assigned forces given in the current configuration $\mathscr{B}_t$ as follows

$$\mathscr{P}_a = \int_{\partial\mathscr{B}_t} (\boldsymbol{n}\cdot\boldsymbol{\sigma}^*)\cdot\boldsymbol{v}\,\mathrm{d}(\partial\mathscr{B}_t) + \int_{\mathscr{B}_t} \rho_t \boldsymbol{b}_t\cdot\boldsymbol{v}\,\mathrm{d}\mathscr{B}_t, \tag{98}$$

where $\boldsymbol{n}$ designates the normal to the boundary $\partial\mathscr{B}_t$, $\boldsymbol{\sigma}^*$ the assigned traction on the boundary $\partial\mathscr{B}_t$, $\boldsymbol{v}$ the velocity field on the current configuration $\mathscr{B}_t$, ρ_t the mass density in $\mathscr{B}_t$, and $\boldsymbol{b}_t$ is the body force assigned on $\mathscr{B}_t$, with $\mathrm{d}\mathscr{B}_t = j_t\mathrm{d}\mathscr{B}$. The expression for $\mathscr{P}_a$ can be recast in the material configuration as follows

$$\mathscr{P}_a = \int_{\partial\mathscr{B}} (\mathfrak{N}\cdot\boldsymbol{P}^*)\cdot\boldsymbol{V}\,\mathrm{d}(\partial\mathscr{B}) + \int_{\mathscr{B}} \rho\boldsymbol{B}_t\cdot\boldsymbol{V}\,\mathrm{d}\mathscr{B} =: {}^{[\mathrm{T}]}\mathscr{P}_a + {}^{[\boldsymbol{B}]}\mathscr{P}_a, \tag{99}$$

where $\mathfrak{N}$ is the normal on the material boundary $\partial\mathscr{B}$, $\boldsymbol{P}^*$ the first Piola–Kirchhoff stress tensor corresponding to $\boldsymbol{\sigma}^*$, $\boldsymbol{V}$ the material velocity field as defined in Eq. $(79)_1$, $\rho = \rho_t j_t$ the mass density in the material configuration $\mathscr{B}$, and $\boldsymbol{B}_t = \boldsymbol{b}_t \circ \boldsymbol{\Phi}_t$ the body force expressed in $\mathscr{B}$, i.e., $\boldsymbol{B}_t(\boldsymbol{\xi}) = \boldsymbol{b}_t(\boldsymbol{\Phi}_t(\boldsymbol{\xi}))$. In Eq. (99), ${}^{[\mathrm{T}]}\mathscr{P}_a$ denotes the power of the assigned Traction forces, and ${}^{[\mathrm{B}]}\mathscr{P}_a$ the power of the assigned Body forces. Note that we have the following decomposition of the material configuration

$$\mathscr{B} = \mathscr{A}\times\mathscr{H}, \quad \partial\mathscr{B} = S\cup\mathscr{A}^+\cup\mathscr{A}^-, \tag{100}$$

as already mentioned in Eq. $(8)_2$, with the lateral surface $\mathscr{S}$ defined in Eq. (10), and where the top facet $\mathscr{A}^+$ and the bottom facet $\mathscr{A}^-$ of $\mathscr{B}$ are defined in a succinct manner as

$$\mathscr{A}^s := \left\{\xi = \left(\xi^\alpha, \xi^3\right) \in \mathscr{B} \,|\, \xi^\alpha \in \mathscr{A}, \xi^3 = sH^s\right\}, \tag{101}$$

with H^s being the distance from the reference surface to the top or bottom facet, i.e.,

$$H^s := {}_{(s\mathbb{N})}Z + {}_{(s\mathbb{N})}h^s > 0, \tag{102}$$

where $s\mathbb{N}$ was defined in Eq. (69).

The integral in the power ${}^{[\mathrm{T}]}\mathscr{P}_a$ due to assigned traction force on the boundary $\partial\mathscr{B}$ can be decomposed into three integrals, according to Eq. $(100)_2$, as follows

$${}^{[T]}\mathscr{P}_a = {}^{[T1]}\mathscr{P}_a + {}^{[T2+]}\mathscr{P}_a + {}^{[T2-]}\mathscr{P}_a, \tag{103}$$

where ${}^{[T1]}\mathscr{P}_a$ is the contribution from the lateral surface boundary $\mathscr{S}$, ${}^{[T2+]}\mathscr{P}_a$ from the top facet $\mathscr{A}^+$, and ${}^{[T2-]}\mathscr{P}_a$ from the bottom facet $\mathscr{A}^-$. Using Eqs. (9) and (10), we can write ${}^{[T1]}\mathscr{P}_a$ as follows

$${}^{[T1]}\mathscr{P}_a = \int_{\partial\mathscr{A}} \sum_{\ell\in\underline{\mathscr{N}}} \int_{(\ell)\mathscr{H}} \left({}_{(\ell)}\mathfrak{N}\cdot{}_{(\ell)}\boldsymbol{P}^*\right)\cdot{}_{(\ell)}\boldsymbol{V}\,\mathrm{d}\xi^3\,\mathrm{d}(\partial\mathscr{A}). \tag{104}$$

It can be seen from the definition of the material velocity ${}_{(\ell)}\boldsymbol{V}$ in Eqs. $(79)_1$ and (80) that ${}_{(\ell)}\boldsymbol{V}$ depends on the transverse material coordinate ξ^3. Let us introduce the following definition of resultant force and resultant couple

$$ {}^{[T1]}_{(\ell)}\boldsymbol{n}^* := \int_{{}_{(\ell)}\mathscr{H}} {}_{(\ell)}\mathfrak{R}\cdot{}_{(\ell)}\boldsymbol{P}^* \,\mathbf{d}\xi^3, \quad {}^{[T1]}_{(\ell)}\boldsymbol{m}^* := \int_{{}_{(\ell)}\mathscr{H}} \left({}_{(\ell)}\mathfrak{R}\cdot{}_{(\ell)}\boldsymbol{P}^*\right)\xi^3 \,\mathbf{d}\xi^3. \tag{105} $$

Next, using Eqs. $(79)_1$, (80), (82) and (83), and Lemma 3.1, we can write ${}^{[T1]}\mathscr{P}_a$ in Eq. (104) as follows

$$ \begin{aligned} {}^{[T1]}\mathscr{P}_a = {} & \int_{\partial\mathscr{A}} \left(\sum_{\ell\in\underline{\mathscr{N}}} {}^{[T1]}_{(\ell)}\boldsymbol{n}^*\right)\cdot\dot{\boldsymbol{u}}\, \mathrm{d}(\partial\mathscr{A}) + \int_{\partial\mathscr{A}} \left[{}^{[T1]}_{(0)}\boldsymbol{m}^* + \sum_{s\ell\in\underline{\mathscr{N}}\setminus\{0\}} s_{(0)}h^{+s}\,{}^{[T1]}_{(s\ell)}\boldsymbol{n}^*\right]\cdot{}_{(0)}\dot{\boldsymbol{t}}\, \mathrm{d}(\partial\mathscr{A}) \\ & + \int_{\partial\mathscr{A}} \sum_{s\ell\in\underline{\mathscr{N}}\setminus\{0,\, s\mathbb{N}\}} \left[{}^{[T1]}_{(s\ell)}\boldsymbol{m}^* - s\left({}_{(0)}h^{+s} + \sum_{i=s}^{s(\ell-1)} {}_{(i)}H\right){}^{[T1]}_{(s\ell)}\boldsymbol{n}^* + s_{(s\ell)}H \sum_{i=s(\ell+1)}^{s\mathbb{N}} {}^{[T1]}_{(i)}\boldsymbol{n}^*\right]\cdot{}_{(s\ell)}\dot{\boldsymbol{t}}\, \mathrm{d}(\partial\mathscr{A}) \\ & + \int_{\partial\mathscr{A}} \sum_{s\in\{-1,\,+1\}} \left[{}^{[T1]}_{(s\mathbb{N})}\boldsymbol{m}^* - s\left({}_{(0)}h^{+s} + \sum_{i=s}^{s(\mathbb{N}-1)} {}_{(i)}H\right){}^{[T1]}_{(s\mathbb{N})}\boldsymbol{n}^*\right]\cdot{}_{(s\mathbb{N})}\dot{\boldsymbol{t}}\, \mathrm{d}(\partial\mathscr{A}). \end{aligned} \tag{106} $$

Now, the contribution to the power ${}^{[T]}\mathscr{P}_a$ by the assigned traction on the top facet ($\xi^3 = +H^+$) and on the bottom facet ($\xi^3 = -H^-$) can be written in concise form as follows

$$ {}^{[T2s]}\mathscr{P}_a = \int_{\mathscr{A}} \left[\left({}_{(s\mathbb{N})}\mathfrak{R}\cdot{}_{(s\mathbb{N})}\boldsymbol{P}^*\right)\cdot{}_{(s\mathbb{N})}\boldsymbol{V}\right]\Big|_{\xi^3=sH^s} \mathrm{d}\mathscr{A}. \tag{107} $$

Introducing the following definition

$$ {}^{[T2s]}_{(s\mathbb{N})}\boldsymbol{n}^* := \left({}_{(\ell)}\mathfrak{R}\cdot{}_{(\ell)}\boldsymbol{P}^*\right)\Big|_{\xi^3=sH^s}, \qquad {}^{[T2s]}_{(s\mathbb{N})}\boldsymbol{m}^* := \left(\xi^3\,{}_{(\ell)}\mathfrak{R}\cdot{}_{(\ell)}\boldsymbol{P}^*\right)\Big|_{\xi^3=sH^s}, \tag{108} $$

we can then rewrite Eq. (107) in the following form

$$ {}^{[T2s]}\mathscr{P}_a = \int_{\mathscr{A}} \left\{{}^{[T2s]}_{(s\mathbb{N})}\boldsymbol{n}^*\cdot{}_{(s\mathbb{N})}\dot{\boldsymbol{\psi}} + {}^{[T2s]}_{(s\mathbb{N})}\boldsymbol{m}^*\cdot{}_{(s\mathbb{N})}\dot{\boldsymbol{t}}\right\} \mathrm{d}\mathscr{A}, \tag{109} $$

where ${}_{(s\mathbb{N})}\dot{\boldsymbol{\psi}}$ can be expressed in terms of $\dot{\boldsymbol{u}}$ and ${}_{(i)}\dot{\boldsymbol{t}}$ using Eq. (83). Next, the contribution to the power $\mathscr{P}_a$ by the assigned body force, i.e., ${}^{[B]}\mathscr{P}_a$ in Eq. (99), has exactly the same expression as that for ${}^{[T1]}\mathscr{P}_a$ in Eq. (106), except that the left superscript $[T1]$ should be replaced with $[B]$, and ${}^{[B]}_{(\ell)}\boldsymbol{n}^*$ and ${}^{[B]}_{(\ell)}\boldsymbol{m}^*$ are defined as follows

$$ {}^{[B]}_{(\ell)}\boldsymbol{n}^* := \int_{{}_{(\ell)}\mathscr{H}} \rho\boldsymbol{B}_t \,\mathrm{d}\xi^3, \qquad {}^{[B]}_{(\ell)}\boldsymbol{m}^* := \int_{{}_{(\ell)}\mathscr{H}} \rho\boldsymbol{B}_t\xi^3 \,\mathrm{d}\xi^3. \tag{110} $$

Finally, we collect the terms with the factor $\dot{\boldsymbol{u}}$, and the terms with the factor ${}_{(i)}\dot{\boldsymbol{t}}$ in the combined expression for the power $\mathscr{P}_a$. The result is presented below.

On the boundary $\partial\mathscr{A}$ of the shell, we introduce the assigned force $\boldsymbol{n}^{*\alpha}: \partial\mathscr{A} \times \mathbb{R}_+ \to \mathbb{R}^3$, such that $\boldsymbol{n}^* = \boldsymbol{n}^{*\alpha}{}_{(0)}\nu_\alpha$, and the assigned couple ${}_{(\ell)}\tilde{\boldsymbol{m}}^{*\alpha}: \partial\mathscr{A} \times \mathbb{R}_+ \to \mathbb{R}^3$ for layer (ℓ). In the interior $\mathscr{A}$ of the shell, let $\boldsymbol{n}^*: \mathscr{A} \times \mathbb{R}_+ \to \mathbb{R}^3$ be the distributed assigned force on the centroidal surface of the reference layer (0), and ${}_{(\ell)}\tilde{\boldsymbol{m}}^*: \mathscr{A} \times \mathbb{R}_+ \to \mathbb{R}^3$ the distributed assigned couple on the centroidal surface of layer (ℓ). The power of the assigned forces and couples is then written as

$$\mathscr{P}_a = \oint_{\partial\mathscr{A}} \left[\boldsymbol{n}^{*\alpha} \cdot \dot{\boldsymbol{u}} + \sum_{\ell \in \underline{\mathscr{N}}} {}_{(\ell)}\tilde{\boldsymbol{m}}^{*\alpha} \cdot {}_{(\ell)}\dot{\boldsymbol{t}} \right] {}_{(0)}\nu_\alpha \, \mathrm{d}(\partial\mathscr{A}) + \int_{\mathscr{A}} \left[\boldsymbol{n}^{*} \cdot \dot{\boldsymbol{u}} + \sum_{\ell \in \underline{\mathscr{N}}} {}_{(\ell)}\tilde{\boldsymbol{m}}^{*} \cdot {}_{(\ell)}\dot{\boldsymbol{t}} \right] \mathrm{d}\mathscr{A}, \tag{111}$$

where we have made use of Eq. (11).

Remark 3.10. *Note that the assigned forces and couples* $\boldsymbol{n}^{*\alpha}$, ${}_{(\ell)}\tilde{\boldsymbol{m}}^{*\alpha}$, $\boldsymbol{n}^{*}$, ${}_{(\ell)}\tilde{\boldsymbol{m}}^{*}$ *are spatial tensors similar to the definition of the spatial tensors* ${}_{(\ell)}\boldsymbol{n}^{\alpha}$, ${}_{(s\ell)}\tilde{\boldsymbol{m}}^{\alpha}$ *in* Eqs. (58) and (59). *The integration in* Eq. (111) *is carried out, however, in the material configuration* $\mathscr{B}$.

In terms of the assigned traction and assigned body force, we have the following expression for $\boldsymbol{n}^*$

$$\boldsymbol{n}^* = \boldsymbol{n}^{*\alpha}{}_{(0)}\nu_\alpha = \sum_{\ell \in \underline{\mathscr{N}}} {}^{[T1]}_{(\ell)}\boldsymbol{n}^* \text{ on } \partial\mathscr{A}, \tag{112}$$

$$\boldsymbol{n}^* = \sum_{s \in \{-1, +1\}} {}^{[T2s]}_{(s\mathbb{N})}\boldsymbol{n}^* + \sum_{\ell \in \underline{\mathscr{N}}} {}^{[B]}_{(\ell)}\boldsymbol{n}^* \text{ in } \mathscr{A}. \tag{113}$$

We will give below the expressions for ${}_{(\ell)}\tilde{\boldsymbol{m}}^*$ on $\partial\mathscr{A}$ first and then in $\mathscr{A}$. By noting that ${}_{(\ell)}\tilde{\boldsymbol{m}}^* = {}_{(\ell)}\tilde{\boldsymbol{m}}^{*\alpha}{}_{(0)}\nu_\alpha$ on the boundary $\partial\mathscr{A}$, we obtain from the expression (106), for ${}^{[T1]}\mathscr{P}_a$, the following

$${}_{(0)}\tilde{\boldsymbol{m}}^* = {}^{[T1]}_{(0)}\boldsymbol{m}^* + \sum_{s\ell \in \underline{\mathscr{N}} \setminus \{0\}} s_{(0)}h^{+s} \, {}^{[T1]}_{(s\ell)}\boldsymbol{n}^* \tag{114}$$

for layer (0),

$${}_{(s\ell)}\tilde{\boldsymbol{m}}^* = {}^{[T1]}_{(s\ell)}\boldsymbol{m}^* - s\left({}_{(0)}h^{+s} + \sum_{i=s}^{s(\ell-1)} {}_{(i)}H \right) {}^{[T1]}_{(s\ell)}\boldsymbol{n}^* + s_{(s\ell)}H \sum_{i=s(\ell+1)}^{s\mathbb{N}} {}^{[T1]}_{(i)}\boldsymbol{n}^* \tag{115}$$

for $s\ell \in \underline{\mathscr{N}} \setminus \{0, \mathbb{N}\}$, and

$${}_{(s\mathbb{N})}\tilde{\boldsymbol{m}}^* = {}^{[T1]}_{(s\mathbb{N})}\boldsymbol{m}^* - s\left({}_{(0)}h^{+s} + \sum_{i=s}^{s(\mathbb{N}-1)} {}_{(i)}H \right) {}^{[T1]}_{(s\mathbb{N})}\boldsymbol{n}^* \tag{116}$$

for the top or bottom layer. Next, inside $\mathscr{A}$, the expressions for ${}_{(\ell)}\tilde{\boldsymbol{m}}^*$ come from ${}^{[T2+]}\mathscr{P}_a$, ${}^{[T2-]}\mathscr{P}_a$, and ${}^{[B]}\mathscr{P}_a$. From the expression for ${}^{[T2s]}\mathscr{P}a$ in Eq. (107) (after full expansion in terms of $\dot{\boldsymbol{u}}$ and ${}_{(i)}\dot{\boldsymbol{t}}$ as indicated above) and from the expression for ${}^{[B]}\mathscr{P}_a$ (which is similar to that of ${}^{[T1]}\mathscr{P}_a$), we obtain the following

$${}_{(0)}\tilde{\boldsymbol{m}}^* = \sum_{s \in \{-1, +1\}} s_{(0)}h^{+s} {}^{[T2s]}_{(s\mathbb{N})}\boldsymbol{n}^* + {}^{[B]}_{(0)}\boldsymbol{m}^* + \sum_{s\ell \in \underline{\mathscr{N}} \setminus \{0\}} s_{(0)}h^{+s} {}^{[B]}_{(s\ell)}\boldsymbol{n}^* \tag{117}$$

for layer (0),

$${}_{(s\ell)}\tilde{\boldsymbol{m}}^* = s_{(s\ell)}H {}^{[T2s]}_{(s\mathbb{N})}\boldsymbol{n}^* + {}^{[B]}_{(s\ell)}\boldsymbol{m}^* - s\left({}_{(0)}h^{+s} + \sum_{i=s}^{s(\ell-1)} {}_{(i)}H \right) {}^{[B]}_{(s\ell)}n^* + s_{(s\ell)}H \sum_{i=s(\ell+1)}^{s\mathbb{N}} {}^{[B]}_{(i)}\boldsymbol{n}^* \tag{118}$$

for $s\ell \in \underline{\mathscr{N}} \setminus \{0, s\mathbb{N}\}$, and

$$ {}_{(s\mathbb{N})}\tilde{\boldsymbol{m}}^{*} = {}^{[T2s]}_{(s\mathbb{N})}\boldsymbol{m}^{*} - s\left({}_{(0)}h^{+s} + \sum_{i=s}^{s(\mathbb{N}-1)} {}_{(i)}H\right){}^{[T2s]}_{(s\mathbb{N})}\boldsymbol{n}^{*} + {}^{[B]}_{(s\mathbb{N})}\boldsymbol{m}^{*} - s\left({}_{(0)}h^{+s} + \sum_{i=s}^{s(\mathbb{N}-1)} {}_{(i)}H\right){}^{[B]}_{(s\mathbb{N})}\boldsymbol{n}^{*} \tag{119} $$

for the top or bottom layer.

3.4. Equations of motion in weighted resultants

We are now ready to present the equations of motion in terms of the weighted resultants based on the balance of power

$$ \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{K} + \mathscr{P}_c = \mathscr{P}_a, \tag{120} $$

where the time rate of the kinetic energy $\mathrm{d}\mathscr{K}/\mathrm{d}t$ is given in Eq. (94), the power $\mathscr{P}_c$ of contact forces/couples given in Eq. (75), and the power $\mathscr{P}_a$ of assigned forces/couples given in Eq. (111).

Remark 3.11. *The balance of power* (120) *corresponds to the Theorem of Power Expended in* Gurtin (1981, p. 180), *which states that the power $\mathscr{P}_a$ expended on the body $\mathscr{B}_t$ by the assigned forces/couples is equal to the stress power $\mathscr{P}_c$ plus the rate of change of the kinetic energy* $\mathrm{d}\mathscr{K}/\mathrm{d}t$.

We now transform Eq. (120) into an expression of balance of virtual power, as enunciated in the principle of virtual power below

$$ \mathscr{P}_i^{\diamond} + \mathscr{P}_c^{\diamond} = \mathscr{P}_a^{\diamond}, \tag{121} $$

where $\mathscr{P}_i^{\diamond}$ designates the virtual power of inertia forces/couples, $\mathscr{P}_c^{\diamond}$ the virtual power of contact forces/couples, and $\mathscr{P}_a^{\diamond}$ the virtual power of assigned forces/couples (see, e.g., Germain, 1972; Lemaitre and Chaboche, 1990).[13] Let the symbol $\diamond$ designate the virtual velocity. Thus, $\dot{\boldsymbol{u}}^{\diamond}$ is the virtual velocity of the centroidal surface of the reference layer (0), and ${}_{(\ell)}\dot{\boldsymbol{t}}^{\diamond}$ is the virtual director rate of layer (ℓ). From Eq. (94), the virtual power $\mathscr{P}_i^{\diamond}$ of the inertia forces/couples can be written as

$$ \mathscr{P}_i^{\diamond} = \int_{\mathscr{A}} \left[\mathfrak{f} \cdot \dot{\boldsymbol{u}}^{\diamond} + \sum_{\ell \in \underline{\mathscr{N}}} {}_{(\ell)}\boldsymbol{C} \cdot {}_{(\ell)}\dot{\boldsymbol{t}}^{\diamond} \right] \mathrm{d}\mathscr{A}, \tag{122} $$

Based on Eq. (75), the virtual power $\mathscr{P}_c^{\diamond}$ of internal forces/couples can be written as

$$ \mathscr{P}_c^{\diamond} = \mathscr{L}^{\diamond} + \sum_{\ell \in \underline{\mathscr{N}}} {}_{(\ell)}\aleph^{\diamond}, \tag{123} $$

from Eq. (64), we have linear momentum part written as

$$ \mathscr{L}^{\diamond} = \oint_{\partial\mathscr{A}} \hat{\boldsymbol{n}}^{\alpha} \cdot \dot{\boldsymbol{u}}^{\diamond} {}_{(0)}\nu_{\alpha} \, \mathrm{d}(\partial\mathscr{A}) - \int_{\mathscr{A}} \hat{\boldsymbol{n}}^{\alpha}_{,\alpha} \cdot \dot{u}^{\diamond} \, \mathrm{d}\mathscr{A}, \tag{124} $$

and from Eq. (74), we have the angular momentum part written as

[13] In the French literature, the principle of virtual power is usually presented in the form of $(-\mathscr{P}_c^{\diamond}) + \mathscr{P}_a^{\diamond} = \mathscr{P}_i^{\diamond}$, but with different notations.

$$ {}_{(\ell)}\aleph^{\diamond} = \oint_{\partial\mathscr{A}} {}_{(\ell)}\overline{\mathscr{M}}^{\alpha}\cdot{}_{(\ell)}\dot{\boldsymbol{t}}^{\diamond}\,{}_{(0)}\nu_{\alpha}\,\mathrm{d}(\partial\mathscr{A}) - \int_{\mathscr{A}} {}_{(\ell)}\mathscr{M}\cdot{}_{(\ell)}\dot{\boldsymbol{t}}^{\diamond}\,\mathrm{d}\mathscr{A}. \tag{125} $$

Based on Eq. (111), the virtual power $\mathscr{P}_a^{\diamond}$ of assigned forces/couples can be written as

$$ \mathscr{P}_a^{\diamond} = \oint_{\partial\mathscr{A}} \left[\boldsymbol{n}^{*\alpha}\cdot\dot{\boldsymbol{u}}^{\diamond} + \sum_{\ell\in\underline{\mathscr{N}}} {}_{(\ell)}\tilde{\boldsymbol{m}}^{*\alpha}\cdot{}_{(\ell)}\dot{\boldsymbol{t}}^{\diamond} \right] {}_{(0)}\nu_{\alpha}\,\mathrm{d}(\partial\mathscr{A}) + \int_{\mathscr{A}} \left[\boldsymbol{n}^{*}\cdot\dot{\boldsymbol{u}}^{\diamond} + \sum_{\ell\in\underline{\mathscr{N}}} {}_{(\ell)}\tilde{\boldsymbol{m}}^{*}\cdot{}_{(\ell)}\dot{\boldsymbol{t}}^{\diamond} \right] \mathrm{d}\mathscr{A}. \tag{126} $$

Since Eq. (121) holds for all admissible virtual velocity $\dot{\boldsymbol{u}}^{\diamond}$ and director rates ${}_{(\ell)}\dot{\boldsymbol{t}}^{\diamond}$, the following equations of motion are obtained

$$ \hat{\boldsymbol{n}}^{\alpha}_{,\alpha} + \boldsymbol{n}^{*} = \boldsymbol{\daleth} \quad \text{in } \mathscr{A}, $$

$$ {}_{(\ell)}\mathscr{M} + {}_{(\ell)}\tilde{\boldsymbol{m}}^{*} = {}_{(\ell)}\boldsymbol{C} \quad \text{in } \mathscr{A}, \forall\ell\in\underline{\mathscr{N}}, \tag{127} $$

with the following boundary conditions on $\partial\mathscr{A}$

$$ \text{Either} \quad \boldsymbol{u} = \boldsymbol{u}^{*} \quad \text{or} \quad \hat{\boldsymbol{n}}^{\alpha} = \boldsymbol{n}^{*\alpha}, $$

$$ \text{Either} \quad {}_{(\ell)}\boldsymbol{t} = {}_{(\ell)}\boldsymbol{t}^{*} \quad \text{or} \quad {}_{(\ell)}\overline{\mathscr{M}}^{\alpha} = {}_{(\ell)}\tilde{\boldsymbol{m}}^{*}, \quad \forall\ell\in\underline{\mathscr{N}}. \tag{128} $$

Remark 3.12. *Comparing* Eq. (127) to Simo and Fox (1989, Eq. (4.19), p. 284), *we see that* Eq. (127) *is much simpler due to the use of the definition of the weighted resultant forces/couples* (58)–(60), *which do not have the factors* $1/\bar{j}$ *and* $\bar{j}$. *See* Remark 3.2.

Remark 3.13. *The equations of motion* (127) *and the boundary conditions* (128) *for multilayer shells reduce exactly to those for sandwich shells presented in* Vu-Quoc et al. (1997b, Eqs. (84) and (85), p. 2531), *which in turn are shown to reduce exactly to those in* Simo and Fox (1989, Eq. (4.19), p. 284). (*See* Remark 3.6 in Vu-Quoc et al. (1997b, p. 2533).

4. Strain measures, stress power, and constitutive laws

To have the elastic constitutive relations for multilayer shells, we first introduce the layer effective stress resultants (membrane force ${}_{(\ell)}\tilde{n}^{\beta\alpha}$, shear force ${}_{(\ell)}\tilde{q}^{\alpha}$, couple ${}_{(\ell)}\tilde{m}^{\beta\alpha}$), then derive the strain measures conjugate to these resultants. We then postulate the elastic constitutive relations for multilayer shells that relate the stress resultants to the conjugate strain measures.

4.1. Constitutive restriction

The equations of motion (127) can be thought of as the resultant forms of the local balance of linear momentum

$$ \mathrm{div}_{(\ell)}\boldsymbol{\sigma} + {}_{(\ell)}\rho_{t}\,{}_{(\ell)}\boldsymbol{b}_{t} = {}_{(\ell)}\rho_{t}\,{}_{(\ell)}\boldsymbol{a}. \tag{129} $$

We have defined the weighted stress resultants ${}_{(\ell)}\boldsymbol{n}^{\alpha}$, ${}_{(\ell)}\tilde{\boldsymbol{m}}^{\alpha}$, ${}_{(\ell)}\ell$ in Eqs. (58)–(60). These stress resultants are related to each other by the resultant form of the local balance of angular momentum (i.e., the symmetry of the Cauchy stress tensor)

$$ {}_{(\ell)}\boldsymbol{\sigma} = {}_{(\ell)}\boldsymbol{\sigma}^{\mathrm{T}}, \quad \text{or} \quad \left({}_{(\ell)}\boldsymbol{g}^{I} \cdot {}_{(\ell)}\boldsymbol{\sigma}\right) \times {}_{(\ell)}\boldsymbol{g}_{I} = 0. \tag{130} $$

On multiplying Eq. $(130)_2$ by ${}_{(\ell)}j_t$, integrating the result over the layer thickness ${}_{(\ell)}\mathscr{H}$, and making use of Eqs. (30) and (31), we obtain

$$ \int_{(\ell)\mathscr{H}} \left\{ \left({}_{(s\ell)}\boldsymbol{g}^{\alpha} \cdot {}_{(s\ell)}\boldsymbol{\sigma}\right) \times \left[{}_{(s\ell)}\boldsymbol{\varphi}_{,\alpha} + \left(\xi^3 - s_{(s\ell)}Z\right)_{(s\ell)}\boldsymbol{t}_{,\alpha}\right] + \left({}_{(s\ell)}\boldsymbol{g}^{\alpha} \cdot {}_{(s\ell)}\boldsymbol{\sigma}\right) \times {}_{(s\ell)}\boldsymbol{t} \right\}_{(\ell)} j_t \, \mathrm{d}\xi^3 = 0. \tag{131} $$

Substituting in Eq. (131) the definition of ${}_{(\ell)}\boldsymbol{n}^{\alpha}$, ${}_{(\ell)}\tilde{\boldsymbol{m}}^{\alpha}$, ${}_{(\ell)}\ell$ in Eqs. (58)–(60), we obtain the following constrained equation

$$ {}_{(\ell)}\boldsymbol{n}^{\alpha} \times {}_{(\ell)}\boldsymbol{\varphi}_{,\alpha} + {}_{(\ell)}\tilde{\boldsymbol{m}}^{\alpha} \times {}_{(\ell)}\boldsymbol{t}_{,\alpha} + {}_{(\ell)}\boldsymbol{\ell} \times {}_{(\ell)}\boldsymbol{t} = 0, \tag{132} $$

called the constitutive restriction in terms of ${}_{(\ell)}\boldsymbol{n}^{\alpha}$, ${}_{(\ell)}\tilde{\boldsymbol{m}}^{\alpha}$, ${}_{(\ell)}\boldsymbol{\ell}$.

4.2. Balance of angular momentum in true resultant couples

The constitutive restriction (132) can be eliminated from the balance of angular momentum equation $(127)_2$ to obtain an alternative form of the balance of angular momentum equation in terms of the true resultant couples. As noted in Remark 3.2, the weighted resultant couple ${}_{(\ell)}\tilde{\boldsymbol{m}}^{\alpha}$ defined in Eq. (59) is not a true moment. We define below the true resultant couple ${}_{(\ell)}\boldsymbol{m}^{\alpha}$ and the true assigned resultant couple ${}_{(\ell)}\boldsymbol{m}^{*}$

$$ {}_{(\ell)}\boldsymbol{m}^{\alpha} := {}_{(\ell)}\boldsymbol{t} \times {}_{(\ell)}\tilde{\boldsymbol{m}}^{\alpha}, \qquad {}_{(\ell)}\boldsymbol{m}^{*} := {}_{(\ell)}\boldsymbol{t} \times {}_{(\ell)}\tilde{\boldsymbol{m}}^{*}. \tag{133} $$

First, consider the reference layer (0). Using the definition of ${}_{(0)}\mathscr{M}$ in Eqs. (67) and (68), we obtain from Eq. $(127)_2$ the following expression for the balance of angular momentum for layer (0)

$$ \left(\mathfrak{M}^{\alpha} + {}_{(0)}\tilde{\boldsymbol{m}}^{\alpha}\right)_{,\alpha} - {}_{(0)}\boldsymbol{\ell} + {}_{(0)}\tilde{\boldsymbol{m}}^{*} = {}_{(0)}\boldsymbol{C}. \tag{134} $$

Now, by taking the cross product between the director ${}_{(0)}\boldsymbol{t}$ and Eq. (134) throughout, and by noting that

$$ {}_{(\ell)}\boldsymbol{t} \times {}_{(\ell)}\tilde{\boldsymbol{m}}^{\alpha}_{,\alpha} = \left({}_{(\ell)}\boldsymbol{t} \times {}_{(\ell)}\tilde{\boldsymbol{m}}^{\alpha}\right)_{,\alpha} - {}_{(\ell)}\boldsymbol{t}_{,\alpha} \times {}_{(\ell)}\tilde{\boldsymbol{m}}^{\alpha}, \tag{135} $$

we can use the constitutive restriction (132) to eliminate the terms ${}_{(0)}\boldsymbol{t}_{,\alpha} \times {}_{(0)}\tilde{\boldsymbol{m}}^{\alpha}$ and ${}_{(0)}\boldsymbol{t} \times {}_{(0)}\ell$ to obtain the following balance of angular momentum for the reference layer (0) in terms of the true couples

$$ {}_{(0)}\boldsymbol{m}^{\alpha}_{,\alpha} + \underbrace{{}_{(0)}\boldsymbol{t} \times \mathfrak{M}^{\alpha}_{,\alpha}} + {}_{(0)}\boldsymbol{\varphi}_{,\alpha} \times {}_{(0)}\boldsymbol{n}^{\alpha} + {}_{(0)}\boldsymbol{m}^{*} = {}_{(0)}\boldsymbol{t} \times {}_{(0)}\boldsymbol{C}. \tag{136} $$

Similarly, using the definition of ${}_{(\ell)}\mathscr{M}$ in Eqs. (73) and (74), we obtain from Eq. $(127)_2$ the following expression for the balance of angular momentum for layer $(s\ell)$

$$ \left[s\left({}_{(s\ell)}h^{-s}{}_{(s\ell)}\boldsymbol{n}^{\alpha} + {}_{(s\ell)}H_{(s\ell)}\mathfrak{N}^{\alpha}\right) + {}_{(s\ell)}\tilde{\boldsymbol{m}}^{\alpha}\right]_{,\alpha} - {}_{(s\ell)}\boldsymbol{\ell} + {}_{(s\ell)}\tilde{\boldsymbol{m}}^{*} = {}_{(s\ell)}\boldsymbol{C}. \tag{137} $$

By taking the cross product of ${}_{(\ell)}\boldsymbol{t}$ with Eq. (137), and by using Eq. (135) together with the constitutive restriction (132) to eliminate the terms ${}_{(s\ell)}\boldsymbol{t}_{,\alpha} \times {}_{(s\ell)}\tilde{\boldsymbol{m}}^{\alpha}$ and ${}_{(s\ell)}\boldsymbol{t} \times {}_{(s\ell)}\ell$ to obtain the following balance of angular momentum for the reference layer $(s\ell)$ in terms of the true couples

$$ {}_{(s\ell)}\boldsymbol{m}^{\alpha}_{,\alpha} + \underbrace{s_{(s\ell)}\boldsymbol{t} \times \left({}_{((s\ell)}h^{-s}{}_{(s\ell)}\boldsymbol{n}^{\alpha} + {}_{(s\ell)}H_{(s\ell)}\mathfrak{N}^{\alpha} \right)_{,\alpha}} + {}_{(s\ell)}\boldsymbol{\varphi}_{,\alpha} \times {}_{(s\ell)}\boldsymbol{n}^{\alpha} + {}_{(s\ell)}\boldsymbol{m}^{*} = {}_{(s\ell)}\boldsymbol{t} \times {}_{(s\ell)}\boldsymbol{C}, \quad \forall s\ell \in \underline{\mathscr{N}}\backslash\{0\}. \tag{138} $$

Remark 4.1. *The true resultant couple* ${}_{(\ell)}\boldsymbol{m}^{\alpha}$ *defined above in Eq.* $(133)_1$ *is the same as the "weighted surface tensor"* $\mathbf{M}_{\alpha}$ *in* Green and Zerna (1968, p. 375, Eq. $(10.2.8)_2$) *whose definition is reproduced below*

$$ \mathbf{M}_{\alpha} = \int_{-(1/2)t}^{(1/2)t} (\mathbf{a}_3 \times \mathbf{T}_{\alpha})\theta_3 \, \mathrm{d}\theta^3, $$

where the tensor $\mathbf{T}_{\alpha}$ *was explained in* Remark 3.2. *The definition of* $\boldsymbol{m}^{\alpha}$ *in* Simo and Fox (1989, Eq. (4.9)), *which differs from that of* ${}_{(\ell)}\boldsymbol{m}^{\alpha}$ *in* Eq. $(133)_1$ *by a factor* $1/\bar{j}$, *corresponds to the quantity* $\mathbf{M}_{\alpha}/\sqrt{a}$ *used in the definition of the "stress couple"* $\mathbf{m}$ *in* Green and Zerna (1968, p. 377, Eq. $(10.2.8)_2$) *as reproduced below*

$$ \mathbf{m} = \frac{v_{\alpha}\mathbf{M}_{\alpha}}{\sqrt{a}}. $$

See Remark 3.2 *for additional details and on the definitions of the resultant forces.*

Remark 4.2. *The balance of angular momentum equations* (136) and (138) *reduce exactly to those for sandwich shells developed in* Vu-Quoc et al. (1997b, Eqs. (155)–(157)).

Remark 4.3. Meaning of the coupling (underbraced) terms. *Without the underbraced terms in* Eqs. (136) and (138), *which comes from the mechanical coupling among the layers, these equations have the mathematical structure of the balance of angular momentum for each individual layer considered separately as a single-layer shell. We note that, for the static case and for the case where* ${}_{(0)}\boldsymbol{m}^{*} = 0$, *the balance of angular momentum for a single-layer shell is similar to* Green and Zerna (1968, p. 380, Eq. (10.4.15)). *Also, note that in the static case, the balance of linear momentum* $(127)_1$ *is similar to* Green and Zerna (1968, p. 380, Eq. (10.4.12)). *We give an idea of what the coupling (underbraced) terms in* Eqs. (136) and (138) *represent physically by considering these terms under a slightly different form. Instead of the actual coupling terms, we consider the following the terms*

$$ {}_{(0)}\boldsymbol{t} \times \mathfrak{M}^{\alpha}, \tag{139} $$

and

$$ s_{(s\ell)}\boldsymbol{t} \times \left({}_{((s\ell)}h^{-s}{}_{(s\ell)}\boldsymbol{n}^{\alpha} + {}_{(s\ell)}H_{(s\ell)}\mathfrak{N}^{\alpha} \right), \tag{140} $$

i.e., without the derivative. The physical interpretation of Eqs. (139) and (140) *is given in* Fig. 6, *by recalling the definition of* $\mathfrak{M}^{\alpha}$ *given in* Eq. (66) *for the left figure, and by considering* $s = +1$ (*i.e. upper layer*) *in* Eq. (140) *for the right figure. In* Fig. 6, ${}_{(\ell)}\mathscr{F}$ *denotes the sum of the layer resultant forces* ${}_{(i)}\boldsymbol{n}^{\alpha}$ *for all layers below layer* (ℓ), *i.e., for i going downward from* $(\ell - 1)$ *to* $(-\check{\mathscr{N}})$. *Now regarding the derivative in the coupling terms as existed in the balance of angular momentum equations* (136) and (138), *we need to consider the equilibrium of an infinitesimal element of the multilayer shell. In our future publications, we will present the complete derivation of the equations of motion from an equilibrium consideration, as opposed to an approach based on the principle of virtual power* (*or calculus of variation*) (*see* Vu-Quoc and Ebcioğlu (2000d)).

4.3. Computational form of stress power and constitutive laws

We summarize here the various expressions that are directly employed in a computational setting for multilayer geometrically-exact shells. For sandwich shells, we refer the readers to Vu-Quoc et al. (1997b) for the derivation of these expressions, which can be easily extended to the multilayer case, as given below.

For computation, the stress power $\mathscr{P}_c$ in Eq. (61) can be recast into the following form[14]

$$\mathscr{P}_c = \sum_{\ell \in \underline{\mathscr{N}}} \int_{\mathscr{A}} \left[{}_{(\ell)}\tilde{n}^{\beta\alpha}{}_{(\ell)}\dot{\epsilon}_{\alpha\beta} + {}_{(\ell)}\tilde{m}^{\beta\alpha}{}_{(\ell)}\dot{\rho}_{\alpha\beta} + {}_{(\ell)}\tilde{q}^{\alpha}{}_{(\ell)}\dot{\delta}_{\alpha} \right] \mathrm{d}\mathscr{A}. \tag{141}$$

The quantities in the integrand of Eq. (141) will be defined and explained below: first the force quantities, followed by the conjugate strain measures.

The effective resultant membrane stress ${}_{(\ell)}\tilde{n}^{\beta\alpha}$ is defined as follows

$${}_{(\ell)}\tilde{n}^{\alpha\beta} := {}_{(\ell)}n^{\alpha\beta} + {}_{(\ell)}\lambda^{\alpha}_{.\mu(\ell)}\tilde{m}^{\mu\beta} = {}_{(\ell)}n^{\alpha\beta} - {}_{(\ell)}\lambda^{\beta}_{.\mu(\ell)}\tilde{m}^{\mu\alpha}, \tag{142}$$

where ${}_{(\ell)}n^{\alpha\beta}$ and ${}_{(\ell)}\tilde{m}^{\mu\beta}$ are the components of the weighted resultant force ${}_{(\ell)}\boldsymbol{n}^{\alpha}$ and the weighted resultant couple ${}_{(\ell)}\tilde{\boldsymbol{m}}^{\mu}$ (defined in Eqs. (58) and (59)), respectively, along the basis vector ${}_{(\ell)}\boldsymbol{\varphi}_{,\beta}$ (see Eqs. (32)–(34)):

$$\{{}_{(\ell)}\boldsymbol{a}_1, {}_{(\ell)}\boldsymbol{a}_2, {}_{(\ell)}\boldsymbol{a}_3\} := \left\{{}_{(\ell)}\boldsymbol{\varphi}_{,1}, {}_{(\ell)}\boldsymbol{\varphi}_{,2}, {}_{(\ell)}\boldsymbol{t}\right\}, \quad \forall \ell \in \underline{\mathscr{N}}. \tag{143}$$

$${}_{(\ell)}\boldsymbol{n}^{\alpha} = {}_{(\ell)}n^{\alpha\beta}{}_{(\ell)}\boldsymbol{a}_{\beta} + {}_{(\ell)}q^{\alpha}{}_{(\ell)}\boldsymbol{a}_3, \tag{144}$$

$${}_{(\ell)}\tilde{\boldsymbol{m}}^{\alpha} = {}_{(\ell)}\tilde{m}^{\alpha\beta}{}_{(\ell)}\boldsymbol{a}_{\beta} + {}_{(\ell)}\tilde{m}^{\alpha 3}{}_{(\ell)}\boldsymbol{a}_3. \tag{145}$$

The quantity ${}_{(\ell)}\lambda^{\alpha}_{.\mu}$ in Eq. (142) is the component of the vector ${}_{(\ell)}\boldsymbol{t}_{,\alpha}$ along the basis vector ${}_{(\ell)}\boldsymbol{\varphi}_{,\mu}$

$${}_{(\ell)}\boldsymbol{t}_{,\alpha} = {}_{(\ell)}\lambda^{\mu}_{.\alpha(\ell)}\boldsymbol{a}_{\mu} + {}_{(\ell)}\lambda^{3}_{.\alpha(\ell)}\boldsymbol{a}_3. \tag{146}$$

The effective transverse shear force ${}_{(\ell)}\tilde{q}^{\alpha}$ is then defined as[15]

$${}_{(\ell)}\tilde{q}^{\alpha} := {}_{(\ell)}q^{\alpha} - {}_{(\ell)}\lambda^{3}_{.\mu(\ell)}\tilde{m}^{\mu\alpha}, \tag{147}$$

where ${}_{(\ell)}q^{\alpha}$, ${}_{(\ell)}\tilde{m}^{\alpha\mu}$, and ${}_{(\ell)}\lambda^{3}_{.\mu}$ come from Eqs. (144)–(146), respectively.

Remark 4.4. *We note the following symmetry conditions for the membrane force* ${}_{(\ell)}\tilde{n}^{\alpha\beta}$ *and bending moment* ${}_{(\ell)}\tilde{m}^{\alpha\beta}$

[14] The derivation of Eq. (141) includes the elimination of the unitary constraints of the directors, i.e., $\|{}_{(\ell)}\boldsymbol{t}\| = 1$. See Vu-Quoc et al. (1997b, p. 2543, Eq. (161)).

[15] There are some misprints in Eq. (174), p. 2544, in Vu-Quoc et al. (1997b): The order of the superscripts ($\mu\alpha$) in ${}_{(\ell)}\tilde{m}^{\mu\alpha}$ should be as shown in Eq. (147), because of our definition of the moments, and not ${}_{(\ell)}\tilde{m}^{\alpha\mu}$.

$$ {}_{(\ell)}\tilde{n}^{\alpha\beta} = {}_{(\ell)}\tilde{n}^{\beta\alpha}, \tag{148} $$

$$ {}_{(\ell)}\tilde{m}^{\alpha\beta} = {}_{(\ell)}\tilde{m}^{\beta\alpha}, \tag{149} $$

While the symmetry of the membrane force in Eq. (148) *can be proven* (see Vu-Quoc et al. (1997b, Eq. (170)), *the symmetry of the bending moment in* Eq. (149) *is an assumption, following* Simo and Fox (1989).

Now, we define the conjugate strain measures. Let the initial value of $\{{}_{(\ell)}\boldsymbol{a}_i\}$ be written as $\{{}_{(\ell)}\boldsymbol{A}_i\}$, i.e.,

$$ \left\{{}_{(\ell)}\boldsymbol{A}_1, {}_{(\ell)}\boldsymbol{A}_2, {}_{(\ell)}\boldsymbol{A}_3\right\} := \{{}_{(\ell)}\boldsymbol{a}_1, {}_{(\ell)}\boldsymbol{a}_2, {}_{(\ell)}\boldsymbol{a}_3\}|_{t=0}, \quad \forall \ell \in \underline{\mathcal{N}}. \tag{150} $$

The membrane strain ${}_{(\ell)}\epsilon_{\alpha\beta}$ and the bending strain ${}_{(\ell)}\rho_{\alpha\beta}$ conjugated with the membrane force ${}_{(\ell)}\tilde{n}^{\beta\alpha}$ and the bending moment ${}_{(\ell)}\tilde{m}^{\beta\alpha}$, respectively, can now be defined as follows:

Membrane strain ${}_{(\ell)}\epsilon_{\alpha\beta}$:

$$ {}_{(\ell)}\epsilon_{\alpha\beta} := \frac{1}{2}\left({}_{(\ell)}a_{\alpha\beta} - {}_{(\ell)}A_{\alpha\beta}\right), \tag{151} $$

$$ {}_{(\ell)}a_{\alpha\beta} := {}_{(\ell)}\boldsymbol{a}_\alpha \cdot {}_{(\ell)}\boldsymbol{a}_\beta, \qquad {}_{(\ell)}A_{\alpha\beta} := {}_{(\ell)}\boldsymbol{A}_\alpha \cdot {}_{(\ell)}\boldsymbol{A}_\beta. \tag{152} $$

Bending strain ${}_{(\ell)}\rho_{\alpha\beta}$:

$$ {}_{(\ell)}\rho_{\alpha\beta} = {}_{(\ell)}\kappa_{\alpha\beta} - {}_{(\ell)}K_{\alpha\beta}, \tag{153} $$

$$ {}_{(\ell)}\kappa_{\alpha\beta} := {}_{(\ell)}\boldsymbol{a}_\alpha \cdot {}_{(\ell)}\boldsymbol{a}_{3,\,\beta}, \qquad {}_{(\ell)}K_{\alpha\beta} := {}_{(\ell)}\boldsymbol{A}_\alpha \cdot {}_{(\ell)}\boldsymbol{A}_{3,\,\beta}. \tag{154} $$

Shear strain ${}_{(\ell)}\delta_\alpha$:

$$ {}_{(\ell)}\delta_\alpha \equiv {}_{(\ell)}\gamma_\alpha - {}_{(\ell)}\Gamma_\alpha, \tag{155} $$

$$ {}_{(\ell)}\gamma_\alpha = {}_{(\ell)}\boldsymbol{a}_\alpha \cdot {}_{(\ell)}\boldsymbol{a}_3, \qquad {}_{(\ell)}\Gamma_\alpha = {}_{(\ell)}\boldsymbol{A}_\alpha \cdot {}_{(\ell)}\boldsymbol{A}_3. \tag{156} $$

Finally, the linear constitutive laws related the effective resultant forces ${}_{(\ell)}\tilde{n}^{\beta\alpha}$, ${}_{(\ell)}\tilde{m}^{\beta\alpha}$, ${}_{(\ell)}\tilde{q}^{\alpha}$, to their respective conjugate strain measures ${}_{(\ell)}\epsilon_{\alpha\beta}$, ${}_{(\ell)}\rho_{\alpha\beta}$, ${}_{(\ell)}\delta_\alpha$ are given below:

$$ {}_{(\ell)}\tilde{n}^{\alpha\beta} = {}_{(\ell)}\bar{j}_t \frac{{}_{(\ell)}E\,{}_{(\ell)}H}{1 - ({}_{(\ell)}\nu)^2}\,{}_{(\ell)}H^{\beta\alpha\gamma\delta}\,{}_{(\ell)}\epsilon_{\gamma\delta}, \tag{157} $$

$$ {}_{(\ell)}\tilde{m}^{\alpha\beta} = {}_{(\ell)}\bar{j}_t \frac{{}_{(\ell)}E\left({}_{(\ell)}H\right)^3}{12\left[1 - ({}_{(\ell)}\nu)^2\right]}\,{}_{(\ell)}H^{\beta\alpha\gamma\delta}\,{}_{(\ell)}\rho_{\gamma\delta}, \tag{158} $$

$$ {}_{(\ell)}\tilde{q}^{\alpha} = {}_{(\ell)}\bar{j}_t\, \mathrm{k}\, {}_{(\ell)}G\,{}_{(\ell)}H\,{}_{(\ell)}A^{\alpha\beta}\,{}_{(\ell)}\delta_\beta, \tag{159} $$

where ${}_{(\ell)}\bar{j}_t$ is defined in Eq. (62), and

$$ {}_{(\ell)}H := {}_{(\ell)}h^+ + {}_{(\ell)}h^-, \qquad {}_{(\ell)}A^{\alpha\beta} := {}_{(\ell)}\boldsymbol{A}^\alpha \cdot {}_{(\ell)}\boldsymbol{A}^\beta \tag{160} $$

are, respectively, the total thickness of layer (ℓ) and the dual metric tensor of layer (ℓ) in the reference configuration; ${}_{(\ell)}E$, ${}_{(\ell)}G$, ${}_{(\ell)}\nu$ are, respectively, the Young's modulus, shear modulus, and Poisson's ratio for layer (ℓ); k is the shear correction coefficient.[16] The elastic constant ${}_{(\ell)}H^{\beta\alpha\gamma\delta}$ given as follows

$$ {}_{(\ell)}H^{\beta\alpha\gamma\delta} = {}_{(\ell)}\nu\,{}_{(\ell)}A^{\beta\alpha}\,{}_{(\ell)}A^{\gamma\delta} + \frac{1}{2}\left(1 - {}_{(\ell)}\nu\right)\left({}_{(\ell)}A^{\beta\gamma}\,{}_{(\ell)}A^{\alpha\delta} + {}_{(\ell)}A^{\beta\delta}\,{}_{(\ell)}A^{\alpha\gamma}\right) \tag{161} $$

is a fourth-order elasticity tensor.

4.4. Reduction to multilayer beam

We now proceed to unify the present formulation for multilayer shells with that for multilayer (plane) beams developed in Vu-Quoc and Ebcioğlu (1996). Specifically, we will show that the equations of motion in Vu-Quoc and Ebcioğlu (1996, Eq. (3.57)) can be deduced from Eq. (127).

To obtain the balance of linear momentum Vu-Quoc and Ebcioğlu (1996, Eq. $(3.57)_1$), we make the following transformation of notation from shells to beams[17]

$$ {}_{(\ell)}\boldsymbol{n}^2 \to {}_{(\ell)}\boldsymbol{n}, \quad \xi^2 \to S, \quad {}_{(s\ell)}\boldsymbol{\varphi} \to {}_{(s\ell)}\boldsymbol{\Phi}_0, \quad {}_{(s\ell)}\boldsymbol{\psi} \to {}_{(\sigma\ell)}\boldsymbol{\varphi}, \quad {}_{(\ell)}\boldsymbol{t} \to {}_{(\ell)}\boldsymbol{t}_2. \tag{162} $$

Then, with ξ^1 and ${}_{(\ell)}\boldsymbol{n}^1$ ignored, Eq. $(127)_1$ reduces to Vu-Quoc and Ebcioğlu (1996, Eq. $(3.57)_1$) by noting that the following relations $(63)_2$, (89), $(86)_1$, (82) and (83) for shells in Section 3 reduce exactly to the corresponding relations (3.19), (3.44), $(3.40)_1$, (3.36) and (3.37) for beams in Vu-Quoc and Ebcioğlu (1996), respectively.

Next, we consider first the static part of the balance of angular momentum. The following additional transformation of notation is used

$$ {}_{(\ell)}\boldsymbol{m}^2 \to {}_{(\ell)}\boldsymbol{m}, \quad {}_{(\ell)}\boldsymbol{m}\cdot\mathbf{e}_3 = {}_{(\ell)}m, \quad {}_{(\ell)}\boldsymbol{m}^{\alpha}_{,\alpha}\cdot\mathbf{e}_3 \to {}_{(\ell)}m_{,s}, \tag{163} $$

$$ \mathfrak{M}^2 \to \mathfrak{M}, \quad \mathfrak{M}\cdot\mathbf{e}_3 = \mathfrak{M}, \tag{164} $$

$$ {}_{(\ell)}\boldsymbol{m}^*\cdot\mathbf{e}_3 \to {}_{(\ell)}\mathfrak{M}. \tag{165} $$

Note that the definition of $\mathfrak{M}^2$ in Eq. (66) is exactly the same as Vu-Quoc and Ebcioğlu (1996, Eq. (3.23)). For layer (ℓ) in a multilayer beam, we have (Vu-Quoc and Ebcioğlu, 1996, Eq. $(3.7)_2$)

$$ {}_{(\ell)}\boldsymbol{t}_{2,\,S} = -{}_{(\ell)}\theta_{,S}\,{}_{(\ell)}\boldsymbol{t}_1, \tag{166} $$

and that $\{{}_{(\ell)}\boldsymbol{t}_1, {}_{(\ell)}\boldsymbol{t}_2, \mathbf{e}_3\}$ form a set of orthonormal vectors.

We consider first layer (0). From the first term in Eq. (136), and using Eq. $(163)_3$, we obtain the first term ${}_{(0)}m_{,S}$ in the integrand in Vu-Quoc and Ebcioğlu (1996), Eq. (3.24). From the last term on the left-hand side of Eq. (136), and using Eq. (165), we obtain the assigned couple ${}_{(0)}\mathfrak{M}$ in Vu-Quoc and Ebcioğlu (1996, Eq. $(3.57)_2$). Now, we examine the second and third terms of Eq. (136), which can be written in beam notation, with ξ^1 and $\mathfrak{M}^1$ ignored, as follows

$$ {}_{(0)}\boldsymbol{t}_2 \times \mathfrak{M}_{,S} + {}_{(0)}\boldsymbol{\Phi}_0 \times {}_{(0)}\mathbf{n} = \left({}_{(0)}\boldsymbol{t}_2 \times \mathfrak{M}\right)_{,S} + {}_{(0)}\theta_{,S}\,{}_{(0)}\boldsymbol{t}_1 \times \mathfrak{M} + {}_{(0)}\boldsymbol{\Phi}_0 \times {}_{(0)}\mathbf{n}, \tag{167} $$

[16] When k = 5/6, the relation (159) is the same as that given in Naghdi (1972, p. 587).

[17] The reader is referred to Vu-Quoc and Ebcioğlu (1996) for the multilayer beam notation.

after an integration by parts and the use of Eq. (166). Projecting the vector relation (167) along the $\mathbf{e}_3$ direction, and noting that

$$\left({}_{(0)}\boldsymbol{t}_2 \times \right)_{,S} \cdot \mathbf{e}_3 = \left({}_{(0)}\boldsymbol{t}_2 \times \mathfrak{M} \cdot \mathbf{e}_3\right)_{,S} = \left(\mathbf{e}_3 \times {}_{(0)}\boldsymbol{t}_2 \cdot \mathfrak{M}\right)_{,S} = -\left({}_{(0)}\boldsymbol{t}_1 \cdot \mathfrak{M}\right)_{,S}, \tag{168}$$

$${}_{(0)}\theta_{,S(0)}\boldsymbol{t}_1 \times \mathfrak{M} \cdot \mathbf{e}_3 = {}_{(0)}\theta_{,S}\mathbf{e}_3 \times {}_{(0)}\boldsymbol{t}_1 \cdot \mathfrak{M} = {}_{(0)}\theta_{,S(0)}\boldsymbol{t}_2 \cdot \mathfrak{M}, \tag{169}$$

we can rewrite Eq. (167) as follows

$$-\left[{}_{(0)}\mathbf{n} \times {}_{(0)}\boldsymbol{\Phi}_0\right] \cdot \mathbf{e}_3 - \left(\mathfrak{M} \cdot {}_{(0)}\boldsymbol{t}_1\right)_{,S} + \left(\mathfrak{M} \cdot {}_{(0)}\boldsymbol{t}_2\right){}_{(0)}\theta_{,S}, \tag{170}$$

which is exactly the last three term in the integrand in Vu-Quoc and Ebcioğlu (1996, Eq. (3.24)). With the above, we have reduce the static part of the balance of angular momentum in layer (0) from shells to beams. Now, we consider the inertia term, i.e., the right-hand side of Eq. (136); recalling Eq. $(162)_5$, and projecting this term on $\mathbf{e}_3$, we obtain

$$\left({}_{(0)}\boldsymbol{t}_2 \times {}_{(0)}\boldsymbol{C}\right) \cdot \mathbf{e}_3 = \mathbf{e}_3 \times {}_{(0)}\boldsymbol{t}_2 \cdot {}_{(0)}\boldsymbol{C} = -{}_{(0)}\boldsymbol{C} \cdot {}_{(0)}\boldsymbol{t}_1 =: {}_{(0)}C, \tag{171}$$

where ${}_{(0)}\boldsymbol{C}$ is given in Eq. (95), and ${}_{(0)}C$ is the inertia for beams in Vu-Quoc and Ebcioğlu (1996, Eq. (3.53)). We have thus reduced the balance of angular momentum for layer (0) for shells to that for beams.

Now for layer (ℓ), as done above, it is easy to see how to obtain the first term ${}_{(\ell)}m_{,S}$ in the integrand in Vu-Quoc and Ebcioğlu (1996, Eq. (3.29)), and the assigned couple ${}_{(\ell)}\mathfrak{M}$ in Vu-Quoc and Ebcioğlu (1996, Eq. $(3.57)_2$). Next, similar to what we did in Eq. (167), using the conversion to beam notation in Eq. (162), we rewrite the second and the third term in Eq. (138), after an integration by parts on the third term and a rearrangement of the terms, as follows

$${}_{(s\ell)}\boldsymbol{\Phi}_{0,S} \times {}_{(s\ell)}\mathbf{n} - \underbrace{s_{(s\ell)}\boldsymbol{t}_{2,S} \times \left({}_{(s\ell)}h^{-s}{}_{(s\ell)}\mathbf{n}\right.} + {}_{(s\ell)}H_{(s\ell)}\mathfrak{N}\Big) + s\left[{}_{(s\ell)}\boldsymbol{t}_2 \times \left({}_{(s\ell)}h^{-s}{}_{(s\ell)}\mathbf{n} + {}_{(s\ell)}H_{(s\ell)}\mathfrak{N}\right)\right]_{,S} \tag{172}$$

The deformation map ${}_{(s\ell)}\boldsymbol{\Phi}_0$ of beam layer $(s\ell)$ is related to the deformation map ${}_{(0)}\boldsymbol{\Phi}_0$ of beam layer (0) via Eqs. (18) and (19), after using the conversion (162), by

$${}_{(s\ell)}\boldsymbol{\Phi}_0 := {}_{(s[\ell-1])}\boldsymbol{\Phi}_0^{+s} + \underbrace{s_{(s\ell)}h^{-s}{}_{(s\ell)}\boldsymbol{t}_2}, \tag{173}$$

$${}_{(s[\ell-1])}\boldsymbol{\Phi}_0^{+s} := {}_{(0)}\boldsymbol{\Phi}_0 + s\left[-{}_{(0)}h^{-s}{}_{(0)}\boldsymbol{t}_2 + \sum_{i=0}^{s(\ell-1)} {}_{(i)}H_{(i)}\boldsymbol{t}_2\right], \tag{174}$$

which are identical to Vu-Quoc and Ebcioğlu (1996, Eqs. (2.6) and (2.7)). Substituting Eqs. (173) and (174) into Eq. (172), noticing that the term underbraced in Eq. (172) is cancelled as a result of the underbraced term in Eq. (173), and using Eq. (166), we obtain

$$\begin{aligned} &{}_{(0)}\boldsymbol{\Phi}_{0,S} \times {}_{(s\ell)}\mathbf{n} + s_{(0)}h^{-s}{}_{(0)}\theta_{,S(0)}\boldsymbol{t}_1 \times {}_{(s\ell)}\mathfrak{N} - s\sum_{i=0}^{s(\ell-1)} {}_{(i)}H_{(i)}\theta_{,S(i)}\boldsymbol{t}_1 \times {}_{(s\ell)}\mathbf{n} \\ &+ s_{(s\ell)}H_{(s\ell)}\theta_{,S(s\ell)}\boldsymbol{t}_1 \times {}_{(s\ell)}\mathfrak{N} + s\left[{}_{(s\ell)}\boldsymbol{t}_2 \times \left({}_{(s\ell)}h^{-s}{}_{(s\ell)}\mathbf{n} + {}_{(s\ell)}H_{(s\ell)}\mathfrak{N}\right)\right]_{,S}. \end{aligned} \tag{175}$$

Next, we project Eq. (175) on $\mathbf{e}_3$, noting that $\mathbf{e}_3$ is independent of the coordinate S and using a similar manipulation as in Eqs. (168) and (169), to obtain

$$-\left[{}_{(s\ell)}\mathbf{n} \times {}_{(0)}\Phi_{0,\,S}\right]\cdot\mathbf{e}_3 + s_{(0)}h^{-s}{}_{(0)}\theta_{,s}\left({}_{(s\ell)}\mathbf{n}\cdot{}_{(0)}\boldsymbol{t}_2\right)$$

$$-s\sum_{i=0}^{s(\ell-1)} {}_{(i)}H_{(i)}\theta_{,S}\left({}_{(s\ell)}\mathbf{n}\cdot{}_{(i)}\boldsymbol{t}_2\right) + s_{(s\ell)}H_{(s\ell)}\theta_{,S}\left({}_{(s\ell)}\mathfrak{N}\cdot{}_{(s\ell)}\boldsymbol{t}_2\right)$$

$$-s\left[\left({}_{(s\ell)}h^{-s}{}_{(s\ell)}\mathbf{n} + {}_{(s\ell)}H_{(s\ell)}\mathfrak{N}\right)\cdot{}_{(s\ell)}\boldsymbol{t}_1\right]_{,S}, \tag{176}$$

which is the same as the remaining terms (after the first term ${}_{(\ell)}m_{,S}$ mentioned above) in Vu-Quoc and Ebcioğlu (1996, Eq. (3.29)) (even though not presented in the same order). Now come the inertia term on the right-hand side of Eq. (138); the same method used to obtain Eq. (171) can be used here, i.e.,

$$\left({}_{(\ell)}\boldsymbol{t}_2 \times {}_{(\ell)}\boldsymbol{C}\right)\cdot\mathbf{e}_3 = \mathbf{e}_3 \times {}_{(\ell)}\boldsymbol{t}_2\cdot{}_{(\ell)}\boldsymbol{C} = -{}_{(\ell)}\boldsymbol{C}\cdot{}_{(\ell)}\boldsymbol{t}_1 =: {}_{(\ell)}C, \quad \forall \ell \in \underline{\mathscr{N}}\setminus\{0\}, \tag{177}$$

where ${}_{(\ell)}\boldsymbol{C}$ is given in Eqs. (96) and (97), and ${}_{(\ell)}C$ is the inertia for beams in Vu-Quoc and Ebcioğlu (1996, Eqs. (3.54) and (3.55)).

5. Closure

Employing the principle of virtual power, we derived the equations of motion of geometrically-exact multilayer shells, whose dynamics is referred to a fixed inertial frame, thus rendering the inertia operator much simpler than that obtained in other types of formulation for multibody dynamics. The present formulation can be employed for the analysis of multilayer shells undergoing large deformation and large overall motion. The continuity of the displacement across the layer boundaries is exactly enforced. Shear deformation is accommodated independently in each layer. The thickness and the side dimensions of each layer are arbitrary, thus allowing for the modeling of an important class of multilayer structures with ply drop-offs. The number of layers is completely unrestricted. An important features of the present formulation is that the reference layer can be chosen arbitrarily among the layers. We also showed that the equations of motion for geometrically-exact multilayer shells reduce exactly to those for geometrically-exact multilayer beams and 1D plates. The balance of power presented in this paper forms the basic weak form that is employed in the finite element formulation for the computation of multilayer shells (see, e.g., Vu-Quoc et al., 2000b,c).

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